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Variational equivalence and generalized absolute continuity

In [2] Kempisty introduced a generalization of Denjoy integral to functions of two variables. We want to discuss the relationship of his concept to the Henstock (generalized Riemann) integral. In particular, we show that, under some natural assumptions, the integral of Kempisty type is less general than the Henstock integral.

In the process, we stress the significance of variational equivalence.

1.1. We start with a space X, which is either $\mathbb{R}$ or $\mathbb{R}^2$, and a class of its subsets Φ (in $\mathbb{R}$ - closed intervals, in $\mathbb{R}^2$ - closed intervals satisfying specific conditions, or triangles etc.), which is assumed to be at least a semiring (nonoverlapping intervals are treated as disjoint). Φ_+ stands for the ring generated by Φ.

Definition. A **derivation base** on X is a class of subsets of the powerset of $X \times \Phi$.

We are following here the language and notation of [7].

The notions of **partition, integral** (here called Henstock integral), **derivative, variation,** a base that has the **partitioning property,** or **local character,** is **filtering down,** are defined just as in [7].

1.2. Definition. We will say that a base B is **compatible with the euclidean topology** on X if for every set G open in that topology there exists a $\beta \in B$ such that $\beta[G] \subset \beta(G)$.

1.3. Definition. Let B be a derivation base. If F_1, $F_2 : X\times\Phi \to \mathbb{R}$, we say that F_1 and F_2 are **variationally equivalent** on $I_0 \in \Phi$ if for every $\varepsilon > 0$ there is a $\beta \in B$ and a superadditive function $\Omega : \Phi \to \mathbb{R}$ (depending on β) such that $\Omega(I_0) \leqq \varepsilon$ and for every $(x, I) \in \beta(I_0)$

(1) $|F_1(x, I) - F_2(x, I)| \leqq \Omega(I)$.

1.4. Theorem. **If a base B has the partitioning property and is filtering down, then the following are equivalent:**

(i) $F : X\times\Phi \to \mathbb{R}$ is Henstock integrable on I_0;

(ii) There exists an additive $H : \Phi \to \mathbb{R}$ such that $V(H - F, B(I_0)) = 0$;

(iii) F is variationally equivalent to an additive $H : \Phi \to \mathbb{R}$.

Proof. Let us note analogous theorems in [1] (theorem 24.1, p. 40), [3] (theorem 4.14, p. 37), and [7] (lemma 4.4, p. 152).

(i) implies (ii). Let $\varepsilon > 0$. Choose a $\beta \in B(I_0)$ so that for each partition $\pi \subset \beta$ of I_0 we have

(2) $\left| \int_{I_0} dF - \sum_{(x,I)\in\pi} F(x, I) \right| \leqq \varepsilon$.

Let $I_1 \subset I_0$ be an element of Φ_+. Let π^1, π^2 be two arbitrary partitions of I_1, contained in β. Since B has the partitioning property and Φ is a semiring, there exist partitions of I_0, $\pi_1, \pi_2 \subset \beta(I_0)$ which extend π^1, π^2, i.e., $\pi^1 = \pi_1(I_1)$, $\pi^2 = \pi_2(I_1)$. We can assume that $\pi_1 - \pi^1 = \pi_2 - \pi^2$.

From (2) we get

(3) $\left| \sum_{(x,I)\in\pi^1} F(x, I) - \sum_{(x,I)\in\pi^2} F(x, I) \right| =$

$\left| \sum_{(x,I)\in\pi_1} F(x, I) - \sum_{(x,I)\in\pi_2} F(x, I) \right| \leqq \varepsilon$.

Set

(4) $H(I) = \int_I dF$ for $I \subset I_0$, $I \in \Phi$.

From (3) we conclude that

(5) $|H(I) - \sum_{(x,I)\in\pi} F(x,I)| \leqq 4\varepsilon$

whenever $\pi \subset \beta$ is a partition of $I \subset I_0$, $I \in \Phi_+$.

Now if I' and I'' are nonoverlapping and contained in I_0, then using (5) we get

(6) $| H(I' \cup I'') - H(I') - H(I'') | \leqq 12\varepsilon,$

so that H is additive.

We will show now that

(7) $V(H - F, B(I_0)) = 0.$

Take β as before, and let $\pi \subset \beta$ be a partition of I_0. Write

(8) $\pi' = \{(x,I) \in \pi : H(I) - F(x,I) \geqq 0\}$, $\pi'' = \pi - \pi'$,

(9) $E' = \sigma(\pi')$, $E'' = \sigma(\pi'')$.

Using (5) we get

(10) $\sum_{(x,I)\in\pi} |H(I) - F(x,I)| =$

$\sum_{(x,I)\in\pi'} |H(I) - F(x,I)| - \sum_{(x,I)\in\pi''} |H(I) - F(x,I)| =$

$|H(E') - \sum_{(x,I)\in\pi'} F(x,I)| + |H(E'') - \sum_{(x,I)\in\pi''} F(x,I)| \leqq 4\varepsilon,$

so that $V(H - F, \beta(I_0)) \leqq 4\varepsilon$, and that proves (7).

(ii) implies (iii). It suffices to set, for any $\beta \in B$,

(11) $\Omega(I) = V(H - F, \beta(I)), \quad I \in \Phi.$

(iii) implies (i). Let $\varepsilon>0$. Choose a $\beta \in B$ and a superadditive Ω as in the definition of the variational equivalence. For any partition $\pi \subset \beta$ we then get

(12) $|H(I_0) - \sum_{(x,I)\in\pi} F(x,I)| = |\sum_{(x,I)\in\pi} H(I) - \sum_{(x,I)\in\pi} F(x,I)| \leqq$

$\sum_{(x,I)\in\pi} |H(I) - F(x,I)| \leqq \sum_{(x,I)\in\pi} \Omega(I) \leqq \Omega(I_0) \leqq \varepsilon.$

This completes the proof.

1.5. Note that the condition $V(F_1 - F_2, B(I_0) = 0$ is in fact another way to define the variational equivalence. The superadditive function of I in 1.3. is the variation $V(F_1 - F_2, \beta(I))$.

2.1. We will consider ourselves with integration of functions of the form

(13) $F(x, I) = f(x)\lambda(x)$,

where λ stands for the volume of I, and $f : I_0 \to \mathbb{R}$ for some $I_0 \in \Phi$.

We will start with a couple of results which apply to derivation bases in general.

2.1. Lemma. **Let B be a base possesing the partitioning property and having the local character. Let $I_0 \in \Phi$. Suppose $H : \Phi \to \mathbb{R}$ is additive and $f : I_0 \to \mathbb{R}$, $F(x, I) = f(x)\lambda(I)$. Define**

(14) $E = \{x : D_B H_\lambda(x) = f(x)\}$.

Then

(15) $V(H - F, B[E]) = 0$,

Proof. Let $\varepsilon > 0$ be arbitrary. For each $x \in E$ there exists a $\beta_x \in B$ such that whenever $(x, I) \in \beta_x[\{x\}]$

(16) $|H(I) - F(x, I)| \leq \varepsilon\lambda(I)$.

Since B has local character, there exists a $\beta \in B$ such that $\beta \subset \beta_x[\{x\}]$ for each $x \in E$. Let π be a partition contained in $\beta[E]$. Then

(17) $\sum_{(x,I)\in\pi} |H(I) - F(x, I)| \leq \varepsilon \sum_{(x,I)\in\pi} \lambda(I) \leq \varepsilon\lambda(I_0)$.

Thus $V(H - F, \beta[E]) \leq \varepsilon\lambda(I_0)$, and this proves (15).

2.3. Exactly as it is done in [7] (lemma 2.4., p.144) for the bases on the real line, it can be shown that also on $\mathbb{R}^2$, for a base B that has local character, the function

(18) $A \to V(G, B[A])$,

where $G : X \times \Phi \to \mathbb{R}$, is an outer measure.

Therefore, by 1.4. and 2.2., the function F of (14) will be proved Henstock integrable to H, if we show that H and F are variationally equivalent on the complement of E.

Since $V(H - F, B[A]) \leq V(H, B[A]) + V(F, B[A])$ for any set A, it is sufficient to prove that both H and F have their variations on A equal to zero.

2.4. Given a derivation base B that has a local character, and a function $G : X \times \Phi \to \mathbb{R}$, the class of all sets A such that

(19) $V(G, B[A]) = 0$,

is a σ-ideal. Let us call it the G **-zero-ideal.**

2.5. Again, exactly the way it is done in [7] (corollary 2.5, page 146) on the real line, one can show that if the base B has local character, then for a point function $f : X \to \mathbb{R}$, if $V(\lambda, B[A]) = 0$ for some set A, then $V(f\lambda, B[A]) = 0$, simply because the domain of f is expressible as a union of a sequence of sets on each of which f is bounded.

2.6. The most common base on the real line is the one whose elements are given as

(20) $\beta_\delta = \{(x, I) : I \subset (x - \delta(x), x + \delta(x))$ and x is an endpoint of $I\}$,

where δ is a positive function on $\mathbb{R}$. We will denote that base by D.

For a function of the form (13), in this case the Henstock integral is the same as the Denjoy-Perron integral.

That fact is the subject of [4]. It is our intention now to simplify the methods of that paper used to show that the Henstock integral includes the Denjoy-Perron integral.

We will present a simple method of proving that a real-valued function H whose arguments are subintervals of $\mathbb{R}$, which is $ACG_\bullet$ (see [6], p. 231), is a Henstock integral (with respect to the base D), of a point function f, which is equal to the derivative of H a.e..

By 2.2., 2.4. and 2.5., it suffices to show that any set A of measure zero such that H is $AC_\bullet$ (see [6] p. 231) on A, is in the H-zero-ideal (the complement of the set E of (15), or rather its apprapriate replacement for $B = D$, is of measure zero, and H is $ACG_\bullet$ on it).

Let A be such a set. Then H is $AC_\bullet$ on the closure A^- of A, as well.

Let $\varepsilon>0$ be arbitrary. Let $\{x_1,x_2,x_3, \cdots\}$ be the sequence of those points of A which are not both side cluster points of A. Since H is continuous, for each $n \in \mathbb{N}$ there is a $\beta_n \in D$ such that

(21) $V(H, \beta_n[\{x_n\}]) < \varepsilon 2^{-n}$.

Since D has local character, there is a $\beta \in D$ with $\beta[\{x_n\}] \subset \beta_n[\{x_n\}]$ for all n. From (21) we get then

(22) $V(H, \beta[\{x_1,x_2,x_3, \cdots\}]) < \varepsilon,$

so that

(23) $V(H, D[\{x_1,x_2,x_3, \cdots\}] = 0.$

Let us take an arbitrary ε again. Let δ be such a positive number that whenever $I_1, I_2, \ldots, I_n$ are nonoverlapping intervals with endpoints in A^-, and

(24) $\sum_{i=1}^{n} \lambda(I_i) < \delta,$

then

(25) $\sum_{i=1}^{n} osc\,(H, I_i) < \varepsilon.$

Let C be an open set containing A such that $\lambda(C) < \delta$ (λ extends naturally to open sets). Write $C = \bigcup_{i \in \mathbb{N}} (a_i, b_i)$.

Let $x \in A - \{x_1, x_2, x_3, \cdots\}$. Then there is an $i \in \mathbb{N}$ such that $x \in (a_i, b_i)$. By replacing, if necessary, a_i by $inf(A^- \cap (a_i, b_i))$ and b_i by $sup(A^- \cap (a_i, b_i))$, we can assume that $a_i \in A^-$, $b_i \in A^-$. Let $\delta(x)$ be a positive number such that

(26) $(x - \delta(x), x + \delta(x)) \subset (a_i, b_i)$,

such a number can be found since x is not isolated on any side in A.

Now let π be a partition contained in $\beta[A - \{x_1, x_2, \cdots\}]$, where β is defined via the function $\delta(x)$ above. Let $[a_{i_1}, b_{i_1}], \ldots, [a_{i_n}, b_{i_n}]$ be those of $[a_i, b_i]$ intervals for which there is an $(x, I) \in \pi$ such that $I \subset [a_i, b_i]$. Then

(27) $\sum_{k=1}^{n} (b_{i_k} - a_{i_k}) < \delta$

so that

(28) $\sum_{(x,I) \in \pi} |H(I)| \leqq \sum_{k=1}^{n} osc(H, [a_{i_k}, b_{i_k}]) < \varepsilon.$

Thus

(29) $V(H, D[A - \{x_1, x_2, \cdots\}]) = 0.$

Combining (23) and (29) completes the proof.

3.1. Definition. We will be saying that a function $H : \Phi \to \mathbb{R}$ is **absolutely continuous in the sense of Kempisty** on a set E (ACK on E), if for every $\varepsilon>0$ there is a $\delta>0$ such that whenever π is a partition contained in $\beta[E]$ for some $\beta \in B$, and such that

(30) $\sum_{(x,I) \in \pi} \lambda(I) < \delta,$

then

(31) $\sum_{(x,I) \in \pi} |H(I)| < \varepsilon.$

H will be termed $ACGK$ if it is additive, B- continuous (see [7] p. 172) and its domain is expressible as a union of a sequence of sets on each of which H is

ACK.

3.2. The above concepts are based on the definitions of Kempisty in [2]. To make that relationship clearer, let us define a base on $\mathbb{R}^2$ which will be denoted by B_ρ.

It is given as follows:

(32) the class Φ consists of intervals I of regularity $r(I)$ (the ratio of width to the length) not less than a fixed positive number ρ;

(33) $\beta \in B_\rho$ if there is a positive function $\delta : \mathbb{R}^2 \to \mathbb{R}$ such that $(x, I) \in \beta$ whenever $x \in I$, $I \subset$ disk $(x, \delta(x))$, $I \in \Phi$.

ACK and $ACGK$ with respect to the base B_ρ are related to what Kempisty defines as ACr and $ACGr$ functions, although they are slightly different. However, Kempisty's definition of ACr simply does not apply to sets contained in degenerate integrals. The difficulties caused by that will be considered by the author elsewhere.

3.3. Definition. A function $f : I_0 \to \mathbb{R}$, where $I_0 \in \Phi$ will be called integrable in the sense of Kempisty if there exists an $ACGK$ function H such that $D_B H_\lambda(x) = f(x)$ a.e. (i.e., except on a set E such that $V(\lambda, B[E]) = 0$), λ stands for the two-dimensional volume.

The definition, just as the one of the $ACGK$ class, is based on the one in [2]. However, Kempisty adds an additional condition of H being Ir (see [2] p. 26). As it turns out, even without that condition, his integral is included in the corresponding Henstock integral.

3.4. Theorem. **Let B be a derivation base on X that has local character, possesses the partitioning property, and is compatible with the euclidean topology.**

Suppose $f : I_0 \to \mathbb{R}$ is integrable in the sense of Kempisty. Then

$F(x, I) = f(x)\lambda(I)$ **is integrable in the sense of Henstock and both integrals coincide.**

Proof. Let K be the set of all $x \in I_0$ where $D_B H_\lambda(x)$ exists but does not equal $f(x)$ or does not exist.

By 2.2., 2.4., and 2.5. it suffices to show that $V(H, B[K]) = 0$.

It is enough to prove that $V(H, B[A]) = 0$, whenever H is ACK on A and A is of measure zero.

Let $\varepsilon>0$ and let δ be the number given by the condition of H being ACK on A. Since A is of measure zero, there exists an open set U containing A, and such that $\lambda(U)<\delta$.

Since B is finer than the euclidean topology, there exists a $\beta \in B$ such that for every $(x, I) \in \beta[U]$, $I \subset U$.

Let π be a partition contained in $\beta[A]$. We have then $x \in A$ for $(x, I) \in \pi$ and $\sum_{(x,I)\in\pi} \lambda(I) < \delta$, so that

(34) $$\sum_{(x,I)\in\pi} |H(I)| < \varepsilon.$$

This shows that $V(H, B[A]) = 0$, as desired.

3.5. An easy argument similar to the one in [3] (p. 22) shows that B_ρ has the partitioning property. It satisfies the other hypotheses of 3.4., as well. Thus 3.4. holds for B_ρ.

It also holds for the base that is used in [3] (p. 21).

The same applies to the newly introduced base in [5], which is somewhat related to B_ρ.

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