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# A STUDY OF FORAN'S CONDITIONS A(N) AND B(N) AND HIS CLASS $\mathcal{F}$

In this paper we show some interesting properties of Foran's class of functions $\mathcal{F}$. Then we introduce a class $\mathcal{E}$ of continuous functions which strictly contains the class $\mathcal{F}$. The basic properties of the new class are established in Chapter V of the present paper.

Let C denote the Cantor ternary set, i.e., $C = \{x \mid x = \sum c_i/3^i$ with $c_i = 0$ or $c_i = 1$ for each $i\}$. Each point $x \in C$ is uniquely represented by $\sum c_i(x)/3^i = 0,c_1(x)c_2(x)\ldots c_i(x)\ldots$ .

Let $\varphi$ be the Cantor ternary function, i.e., $\varphi(x) = \sum c_k(x)/2^{k+1}$, for each $x \in C$. Then $\varphi$ is continuous on C and, by extending $\varphi$ linearly on each interval contiguous to C, one has $\varphi$ defined and continuous on $[0,1]$.

Definition 1. [2] Given a natural number N and a set E, a function f is said to be B(N) on E if there is a number $M < \infty$ such that for any sequence $I_1, \ldots, I_k, \ldots$ of nonoverlapping intervals with $I_k \cap E \neq \emptyset$, there exist intervals $J_{kn}$, $n=1,2,\ldots,N$, such that

$$B(f;E\cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^{N} (I_k \times J_{kn}) \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |J_{kn}| < M.$$

(Here B(f;X) is the graph of f on the set X.)

Definition 2. [2] Given a natural number N and a set E, a function f is said to be A(N) on E if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $I_1, \ldots, I_k, \ldots$ are nonoverlapping intervals with $E\cap I_k \neq \emptyset$ and $\sum|I_k| < \delta$ , then there exist intervals $J_{kn}$, $n = 1,2,\ldots,N$, such that

$$B(f;E\cap \bigcup_k I_k) \subset \bigcup_k \bigcup_{n=1}^{N} (I_k \times J_{kn}) \quad \text{and} \quad \sum_k \sum_{n=1}^{N} |J_{kn}| < \varepsilon .$$

If a function f is continuous and satisfies A(N) on a bounded set E then f satisfies B(N) on E. (See [2],(v),pp.361)

We denote by $V(f;N) = \inf\{M : M$ is given by the fact that f is B(N) on a set $E\}$.

The class $\mathfrak{F}$ (respectively $\mathfrak{B}$ ) consists of all continuous functions F defined on a closed interval I for which there exist a sequence of sets $\{E_n\}$ and natural numbers $\{N_n\}$ such that $I=\bigcup E_n$ and F is $A(N_n)$ (respectively $B(N_n)$) on $E_n$.

Foran shows in [2] that B(N) generalizes bounded variation and A(N) generalizes absolute continuity. But the fact that these generalizations are strict (if the set E is not an interval) does not follow by Foran's paper [2] since the function F constructed there is AC. (We show this in Chapter I.) Also in Chapter I we construct a continuous function which is A(N+1) on a perfect set and which is not B(N) on any portion of this set. Hence $\mathfrak{B}$ strictly generalizes the class BVG and $\mathfrak{F}$ strictly generalizes the class ACG.

## <u>CHAPTER I</u> - <u>Foran's conditions</u> A(N) <u>and</u> A(N+1)

<u>Foran's function</u> F <u>is</u> AC. Let F be the function defined in [2], i.e., define F on C by $F(\sum c_i/3^i) = \sum c_{h(i)}/3^i$ where $h(i) = j_{k-1}$ when $j_{k-1} \leq i < j_k$ and $j_k$ is a strictly increasing sequence of natural numbers with $j_0=0$. Then F is continuous on C and, by extending F linearly on each interval contiguous to C, one has F defined and continuous on $[0,1]$.

Consider an interval I such that $I \cap C \neq \emptyset$ and $1/3^{m+1} \leqslant |I| < 1/3^m$ for a natural number m. Since $|I| < 1/3^m$, there exist $c_1, c_2, \dots, c_m$ such that if $x \in I \cap C$ then $c_i(x) = c_i$, $i=1,2,\dots,m$. Let $A = \sum_{i=1}^{m} c_i/3^i$ and suppose that $h(m) = j_{k-1}$. Let $J = [F(A), F(A + \sum_{i=m+1}^{\infty} 2/3^i)]$. Then $F(I \cap C) \subset J$ and $|J| =$

$$= c_{j_{k-1}}(1/3^{m+1} + \dots + 1/3^{j_k - 1}) + 2(1/3^{j_k} + 1/3^{j_k+1} + \dots) \leqslant$$

$\leqslant 1/3^m \leqslant 3|I|$. Therefore F is AC on C.

Remark 1. If we define a function $F_1$ analogous with F, but $h(i) = j_k$ when $j_{k-1} \leqslant i < j_k$, then $F_1$ is also AC on C.

Theorem 1. *Given a natural number N, there exists a continuous function $G_N$ which is A(N+1) on a perfect set and which is A(N) on no portion of this set.*

Proof. Let $C_{2N+1} = \{x : x = \sum c_i/(2N+1)^i,\ c_i = 0,2,4,\dots,2N$ for each $i\}$ and define $G_N$ on $C_{2N+1}$ by $G_N(\sum c_i/(2N+1)^i) =$

$= \sum c_{j_k}/(2N+1)^{j_{k-1}}$. Then $G_N$ is continuous on $C_{2N+1}$. Let I be an interval, $I \cap C_{2N+1} \neq \emptyset$, such that $1/(2N+1)^{m+1} \leqslant |I| < 1/(2N+1)^m$ for some m. Since $|I| < 1/(2N+1)^m$ there exist $c_1, c_2, \dots, c_m$ such that if $x \in I \cap C_{2N+1}$ then $c_i(x) = c_i$, $i=1,\dots,m$. Let $k \in N$ be such that $j_{k-1} \leqslant m < j_k$ and let $A = \sum_{i=1}^{m} c_i/(2N+1)^i$.

We define $J_i$ as follows:

$$J_i = \left[G_N\left(A + \frac{2i}{(2N+1)^{j_k}}\right),\ G_N\left(A + \frac{2i}{(2N+1)^{j_k}} + \sum_{i=j_k+1}^{\infty} \frac{2N}{(2N+1)^i}\right)\right],$$

$i = 0,1,\dots,N$. Then $G_N(I \cap C_{2N+1}) \subset \bigcup_{i=0}^{N} J_i$ and $|J_i| =$

$= 2N(1/(2N+1)^{j_k} + 1/(2N+1)^{j_{k+1}} + \ldots) \leqslant (2N+1)\cdot|I|$. Therefore $G_N$ is $A(N+1)$ on $C_{2N+1}$.

Let K be a portion of $C_{2N+1}$. Then $K \supset I' \cap C_{2N+1}$, where

$$I' = \left[\sum_{i=1}^{j_k} c_i/(2N+1)^i \;,\; \sum_{i=1}^{j_k} c_i/(2N+1)^i + \sum_{i=j_k+1}^{\infty} 2N/(2N+1)^i\right].$$

Let $\{I_j\}$, $j=1,2,\ldots,(N+1)^{n_k}$, $n_k = j_{k+1}-j_k-1$, be the retained closed intervals from the $(j_{k+1}-1)$-st step in a $(2N+1)$-ary process (analogous to the Cantor ternary process) which are contained in I' and set $I'_j = [a'_j, b'_j]$. We observe that for each $j = 1,2,\ldots,(N+1)^{n_k}$ there exists $1 \leqslant i \leqslant n_k$ and $c_{j_k+1},\ldots,c_{j_k+i}$ such that $a'_j = 0,c_1\ldots c_{j_k+i}$ and $b'_j = a'_j + \sum_{m=j_{k+1}}^{\infty} 2N/(2N+1)^m$. If $G_N(I_j \cap C_{2N+1}) \subset J_{j,1} \cup \ldots \cup J_{j,N}$ then at least one of the intervals $J_{j,m}$, $m=1,2,\ldots,N$ has the measure greater than

$$2\cdot\frac{1}{(2N+1)^{j_k}} - 2N\cdot\left(\frac{1}{(2N+1)^{j_{k+1}}} + \frac{1}{(2N+1)^{j_{k+2}}}\right) > 1/(2N+1)^{j_k}.$$

If $j_{k+1} = 3j_k$ then $\sum_{j=1}^{(N+1)^{n_k}} (G_N(b'_j) - G_N(a'_j)) \xrightarrow[k\to\infty]{} \infty$. Therefore $G_N$ is not $B(N)$ on K.

CHAPTER II - A continuous function in $\mathcal{F}$ which is B(2) on C and which is not A(N) on C for any natural number N.

Remark 2. Let $a,b \in C$, $a < b$. Then there is a natural number $n \in N$ such that $b-a \in [1/3^{n+1}, 1/3^n)$. If $b-a = 1/3^{n+1}$, we have exactly two possibilities: a) There exist $c_1,\ldots,c_{n+1}$ such that

$a = 0,c_1\ldots c_{n+1}000\ldots$ and $b = 0,c_1\ldots c_{n+1}222\ldots$, hence, clearly $[a,b]$ is a retained closed interval from the step $(n+1)$ in Cantor's ternary process; b) there exist $c_1,\ldots,c_n$, such that $a = 0,c_1\ldots c_n0222\ldots$ and $b = 0,c_1\ldots c_n2000\ldots$, hence, clearly $(a,b)$ is an excluded open interval from the step $(n+1)$ in the Cantor ternary process.

If $b-a \in (1/3^{n+1}, 1/3^n)$, then there exist $c_1,\ldots,c_n$ such that $a = 0,c_1\ldots c_n0c_{n+2}(a)\ldots c_{n+k}(a)\ldots$ and

$b = 0,c_1\ldots c_n2c_{n+2}(b)\ldots c_{n+k}(b)\ldots$ .

Lemma. For any strictly increasing sequence of natural numbers $\{j_i\}$, the function $F(x) = \sum c_{j_i}(x)/2^{j_i+1}$, $x \in C$ is $B(2)$ on $C$ and $V(F,2) \leq 1$ on $C$.

Proof. Let $I$ be an interval such that $I \cap C \neq \emptyset$ and let $a = \inf(I\cap C)$, $b = \sup(I \cap C)$. Suppose that $a<b$ and let $n\in N$ such that $1/3^{n+1} \leq b-a < 1/3^n$. If $b-a = 1/3^{n+1}$, then we have the two situations given by the Remark, namely a) and b). In the case a), let $a_1 = 0,c_1\ldots c_{n+1}0222\ldots$ and $b_1 = 0,c_1\ldots c_{n+1}2000\ldots$ . Clearly $F(I\cap C) \subset [F(a),F(a_1)] \cup [F(b_1),F(b)]$ and

$(F(a_1)-F(a)) + (F(b)-F(b_1)) \leq \varphi(b) - \varphi(a) = |\varphi(I)|$.

In the case b), $F(I\cap C) \subset [F(a),F(a)] \cup [F(b),F(b)]$ and

$(F(a)-F(a)) + (F(b)-F(b)) = \varphi(b) - \varphi(a) = 0$.

If $1/3^{n+1} < b-a < 1/3^n$, then using the Remark for this case, let $a_1 = 0,c_1\ldots c_n0222\ldots$ and $b = 0,c_1\ldots c_n2000\ldots$ . Let $a_2$ belong to $[a,a_1]\cap C$ such that $F(a_2) = \inf(F([a,a_1]\cap C))$ and let $b_2 \in [b_1,b]\cap C$ such that $F(b_2) = \sup(F([b_1,b]\cap C))$. Clearly $F(I\cap C) \subset \subset [F(a_2),F(a_1)] \cup [F(b_1),F(b_2)]$ and $(F(a_1)-F(a_2)) + (F(b_2)-F(b_1)) \leq \varphi(b) - \varphi(a) = |\varphi(I)|$ . Therefore $F$ is $B(2)$ on $C$.

Theorem 2. *For any natural number*, $p \geq 1$, *there exists a continuous function* $F_p:[0,1] \longrightarrow [0, 1/2^{p-1})$ *such that*: a) $F_p$ *is* $A(2^p)$ *on* $C$; b) $F_p$ *is not* $A(2^p-1)$ *on* $C$; c) $F_p$ *is* $B(2)$ *on* $C$ *and* $V(F_p,2) \leq 1$ *on* $C$.

Proof. Let $p$ be a natural number and let $\{j_i\}$ be a strictly increasing sequence of natural numbers such that

$$(1) \qquad 3^{j_i+p} < 2^{j_{i+1}-p} \quad \text{and} \quad j_1 = p.$$

For each $x \in C$ let $F_p(x) = \sum_{i=1}^{\infty} \sum_{k=0}^{p-1} c_{j_i+k}(x)/2^{j_i+k+1}$. Then $F_p$ is continuous on $C$ and, by extending $F_p$ linearly on each interval contiguous to $C$, one has $F_p$ defined and continuous on $[0,1]$. Moreover, $F_p:[0,1] \longrightarrow [0, 1/2^{p-1})$.

a) Let $I$ be an interval such that $I \cap C \neq \emptyset$ and choose $n$ so that $1/3^{n+1} \leq |I| < 1/3^n$. Then there exist $c_1,\dots,c_n$ such that $c_i(x) = c_i$, $i=1,2,\dots,n$, for each $x \in I \cap C$. Let $k$ be a natural number such that $j_{k-1} \leq n < j_k$. Suppose that $k > 1$. (For $k=1$ the proof is similar.) Then we have two possibilities: 1) $j_{k-1} \leq n < j_{k-1}+p-1$ ; 2) $j_{k-1}+p-1 \leq n < j_k$. We shall consider each of them separately. 1) Let $j = n-j_{k-1}$. Clearly $j < p-1$. Hence by (1) we have

$$(2) \qquad \frac{1}{3^{n+1}} = \frac{1}{3^{j_{k-1}+j+1}} = \frac{3^{p-j-1}}{3^{j_{k-1}+p}} > \frac{3^{p-j-1}}{2^{j_k-p}} =$$

$$= \frac{2^p \cdot 2^{j-1} \cdot 3^{p-j-1}}{2^{j_k+j-1}} > \frac{2^p}{2^{j_k+j-1}} .$$

Let $A = \{x \in C \mid c_i(x)=c_i$ for $i=1,2,\ldots,n$; $c_i(x)=0$ for each $i \in \{j_{k-1}+p,\ldots,j_k-1\}$ and $i > j_k+j\}$. Let $x_1 < x_2 < \ldots < x_{2^p}$ be the elements of the set A. Let $y_i = x_i + \sum_{t \geq j_k+j+1} 2/2^t$, $i = 1,2,\ldots,2^p$. Clearly $F_p(I \cap C) \subset \bigcup_{i=1}^{2^p} [F_p(x_i),F_p(y_i)]$ and by (2) $\sum_{i=1}^{2^p} (F_p(y_i)-F_p(x_i)) \leq 2^p/2^{j_k+j-1} < 1/3^{n+1} < |I|$.

2) Let $A = \{x \in C \mid c_i(x)=c_i$ for $i=1,2,\ldots,n$; $c_i(x)=0$ for each $i \in \{n+1,\ldots,j_k-1\}$ and $i \geq j_k+p\}$. Let $x_1 < x_2 < \ldots < x_{2^p}$ be the elements of the set A. Let $y_i = x_i + \sum_{t \geq j_k+p} 2/2^t$. Clearly

$F_p(I \cap C) \subset \bigcup_{i=1}^{2^p} [F_p(x_i),F_p(y_i)]$ and by (1) $\sum_{i=1}^{2^p} (F_p(y_i)-F_p(x_i)) \leq$

$$\leq \frac{2^p}{2^{j_{k+1}-1}} = \frac{2}{2^{j_{k+1}-p}} < \frac{2}{3^{j_k+p}} \leq \frac{2}{3^p} \cdot \frac{1}{3^{n+1}} \leq \frac{2}{3^p} \cdot |I| < |I|.$$

By 1),2) and the definition of $A(2^p)$, it follows that $F_p$ is $A(2^p)$ on C.

b) Let $I_q = [a_q,b_q]$, $q = 1,2,\ldots,2^{j_k-1}$, be the retained closed intervals from the step $j_k-1$ in the Cantor ternary process and Let $I_q^i = [a_q^i,b_q^i]$ be the retained closed intervals from the step $j_k-1+p$ in the Cantor ternary process, which are contained in $I_q$. Clearly $(b_q^i,a_q^{i+1})$, $i = 1,2,\ldots,2^p-1$, are the excluded open intervals from the steps $j_k,j_k+1,\ldots,j_k+p-1$ in the Cantor ternary process. By Remark 2 we have

(3) $\quad F_p(C \cap I_q^i) \subset [F_p(a_q^i),F_p(b_q^i)]$, $i=1,2,\ldots,2^p$ and

(4) $F_p(a_q^{i+1}) - F_p(b_q^i) \geq 2/2^{j_k+p} - 2/2^{j_{k+1}}$.

By (3) and (4) it follows that if we cover the set $F_p(C\cap I_q)$ with $2^p-1$ intervals $J_{qj}$, $j=1,2,\dots,2^p-1$, then there exists a $j\in\{1,2,\dots,2^p-1\}$ such that the interval $J_{qj}$ contains the interval $[F_p(b_q^i),F(a_q^{i+1})]$. Hence for any intervals $J_{qj}$, $j = 1,2,\dots,2^p-1$, $q = 1,2,\dots,2^{j_k-1}$, for which

$B(F_p;C)\subset \bigcup_{q=1}^{2^{j_k-1}} \bigcup_{j=1}^{2^p-1}(I_q\times J_{qj})$, we have $\sum_{q=1}^{2^{j_k-1}} \sum_{j=1}^{2^p-1}|J_{qj}|\geq$

$$\geq 2^{j_k-1}\cdot(2/2^{j_k+p} - 2/2^{j_k+p}) \xrightarrow[k\to\infty]{} 1/2^p.$$

c) follows by the above lemma.

Corollary. There exists a continuous function F on [0,1] such that: a) F is B(2) on C; b) $F\in\mathcal{F}$ on [0,1] ; c) F is not A(N) on C for any natural number N.

Proof. Let $x_n,y_n\in C$ be such that: $c_i(x_n) = c_i(y_n) = 2$, $i = 1,2,\dots,n$; $c_i(x_n) = 0$, $i > n$; $c_{n+1}(y_n) = 0$; $c_i(y_n) = 2$, $i> n+1$. Let F be defined as follows: For each $x\in[x_n,y_n]$, $F(x) = (1/2^n)F_p(3^n\cdot(x-x_n))$ and $F(1) = 0$. Extending F linearly to the intervals $(y_n,x_{n+1})$, we have F defined and continuous on [0,1]. Now, the proof follows by the above Theorem 2.

## CHAPTER III - A function in Foran's class $\mathcal{F}$ not a.e. approximately derivable.

It is well known the following theorem of Denjoy-Khintchine:

A function which is measurable and BVG (respectively ACG) on a set is approximately derivable at almost all points of this set. (See [6], pp.222-223.)

In what follows we show that the above theorem is no longer true if VBG (respectively ACG) is replaced by the class $\mathcal{F}$ .

Theorem 3. There is a function in $\mathcal{F}$ that is not a.e. approximately derivable.

Proof. For each $x \in C$ we define two functions $F_1$ and $F_2$ as follows:

$$F_1(x) = \sum c_{2i-1}(x)/4^i \quad \text{and} \quad F_2(x) = (1/2)\sum c_{2i}(x)/4^i \ .$$

Extending $F_1$ and $F_2$ linearly on each interval contiguous to C, we have $F_1$ and $F_2$ defined and continuous on $[0,1]$. We have shown in [1] that $F_1$ and $F_2$ have the following properties: a) $F_1$ and $F_2$ are ordinary differentiable a.e. on $[0,1]$ ; b) $|F_1(C)| = |F_2(C)| = 0$, hence $F_1$ and $F_2$ satisfy Lusin's condition (N) on $[0,1]$ ; c) $F_1$ and $F_2$ satisfy Foran's condition B(2) on C; d) $F_1$ and $F_2$ do not belong to $\mathcal{F}$ .

Using a composition of $F_1$ and $F_2$ with an homeomorphism h we obtain two continuous functions $G_1 = F_1 \circ h$ and $G_2 = F_2 \circ h$ both of which satisfy our theorem. Moreover, $G_1+G_2$ is ordinary differentiable a.e. on $[0,1]$.

Let $a \in (0,1)$ and let P be the perfect set contained in $[0,1]$ defined as follows: $P = \{y : y = \sum d_i((1-a)/2^i + 3a/4^i)$, with $d_i$ taking the values 0 and 1 only$\}$. Each point $y \in P$ is uniquely represented by $\sum d_i(y)((1-a)/2^i + 3a/4^i)$. Clearly $|P| = 1-a$. Let h be the continuous function defined as follows: For each $y \in P$,

$h(y) = \sum d_i(y)(2/3^i)$. Extending h linearly on each interval contiguous to P one has h defined and continuous on $[0,1]$.

Clearly $h(P) = C$. Let $g$, $G_1$ and $G_2$ be continuous functions on $[0,1]$, defined by $g = \varphi \circ h$, $G_1 = F_1 \circ h$ and $G_2 = F_2 \circ h$. Clearly $G_1 + G_2 = g$, for each $y \in P$ $g(y) = \sum d_i(y)/2^i$, and on each interval contiguous to $P$ $g$ is a constant. In fact $g$ is a Lipschitz function with constant $1/(1-a)$. Indeed, let $a_1, b_1 \in P$, $a_1 < b_1$. Then there exists a $k \in N$ such that $a_1$ and $b_1$ have the first $k$ digits after the comma equal. Let

$$S = \sum_{i=k+2}^{\infty} (d_i(b_1) - d_i(a_1)) \cdot (1/4^{i-k-1}). \text{ Clearly } S \in [-1/3,\ 1/3].$$

Now we have $b_1 - a_1 = \sum_{i=1}^{\infty} (d_i(b_1) - d_i(a_1)) \cdot ((1-a)/2^i + 3a/4^i) =$

$$= (1-a) \cdot (g(b_1) - g(a_1)) + 3a/4^{k+1} + (3a/4^{k+1}) \cdot S > (1-a) \cdot (g(b_1) - g(a_1)).$$

Hence

(5) $\qquad g(b_1) - g(a_1) < (b_1 - a_1)/(1-a)\ ,\ a_1, b_1 \in P,\ a_1 < b_1\ .$

We show now that both $G_1$ and $G_2$ satisfy our theorem. Let $a_1, b_1 \in P$, $a_1 < b_1$, $h(a_1) = a'$, $h(b_1) = b'$. By [1] (the proof of Theorem 1), there exist two closed intervals $J_1$ and $J_2$, with $|J_1| + |J_2| \leq \varphi(b') - \varphi(a')$ and $F_2(C \cap [a', b']) \subset J_1 \cup J_2$. By (5) $|G_2(P \cap [a_1, b_1])| = |F_2(C \cap [a', b'])| \leq |J_1| + |J_2| \leq \varphi(b') - \varphi(a')$ $= g(b_1) - g(a_1) < (b_1 - a_1)/(1-a)$. Hence $G_2$ is $A(2)$ on $P$. Since $g$ is $AC$ on $[0,1]$ it follows that $G_1$ is $A(2)$ on $P$. (See [2], (vi), pp. 361.)

We show now that $G_1$ and $G_2$ are not a.e. approximately derivable. Suppose on the contrary that $G_1$ and $G_2$ are approximately derivable at almost all the points of $P$. By [4] (Lemma K: If $F'_{ap}$ exists at every point of a set $E$ and $|F(E)| = 0$, then $F'_{ap}(x) = 0$ at a.e. point $x \in E$) together with $|G_1(P)| = |G_2(P)| =$

$= 0$ it follows that $(G_1)'_{ap}(x) = (G_2)'_{ap}(x) = 0$ a.e. on P. But $G_1' = -G_2'$ on $[0,1]-P$ and consequently $(G_1+G_2)'_{ap} = 0$ a.e. on $[0,1]$. Since $G_1+G_2 = g \in AC$ it follows that g is identically constant on $[0,1]$. Contradiction.

Remark 3. *The functions $G_1$ and $G_2$ are not a.e. preponderantly derivable. Moreover, there exists a subset E of $[0,1]$ with positive measure such that for every point $x \in E$ there corresponds no measurable set $Q(x)$ for which 1) the linear unilateral density of $Q(x)$ at x is positive on at least one side of the point x and 2) $(\overline{G_i})_{Q(x)}(x) < +\infty$ or $(\underline{G_i})_{Q(x)}(x) > -\infty$, $i = 1,2$.*

Proof. Let $X_1^i = \{x \in P$ : there corresponds a measurable set $Q(x)$ for which 1) and 2)$\}$, $i = 1,2$ and let $X_2^i = \{x \in P : (G_i)'_{pr}(x)$ exists and is finite$\}$ and $X_3^i = \{x \in P : (G_i)'_{ap}(x)$ exists and is finite$\}$, $i = 1,2$. Clearly $X_3^i \subset X_2^i \subset X_1^i$ and $X_3^i$ is measurable (see [4],pp.447 and [6],pp.297 a remark on Theorem 10.1). By a theorem of Denjoy-Khintchine (see [6],pp.295-296 the proof of Theorem 10.1) it follows that $|X_1^i - X_3^i| = 0$. Hence $X_1^i$ and $X_2^i$ are also measurable. By the proof of our theorem it follows that $|X_1^i| = |X_2^i| = |X_3^i| < |P| = 1-a$.

## CHAPTER IV - Differentiation and Foran's class of functions $\mathcal{F}$.

Foran proves in [3] the following theorem:

Given two continuous functions $F_1$ and $F_2$ defined on a closed interval I, if $F_1$ is ACG, if $F_2$ satisfies Lusin's condition (N) and if both $F_1$ and $F_2$ are differentiable a.e., with $F_1' = F_2'$ a.e.

on I, then $F_2-F_1$ is identically constant.

We show now that the above theorem is no longer true if ACG is replaced by Foran's class of functions $\mathcal{F}$.

Theorem 4. *There exist an almost everywhere differentiable, continuous function F belonging to $\mathcal{F}$ on $[0,1]$ and a decreasing, unbounded sequence of almost everywhere differentiable, continuous functions $G_s$, satisfying Lusin's condition (N) on $[0,1]$ such that $G_s' = F'$ a.e. on $[0,1]$ and $G_s(0) = F(0) = 0$ for each natural number s, but $G_s - F$ is not identically 0.*

Proof. Let $\{j_k\}$ be a strictly increasing sequence of natural numbers such that $j_0 = 0$ and

(6) $\quad 3^{j_k} < 2^{j_{k+1}}$, and set $n_k = j_{k+1} - j_k - 1$.

For each $x \in C$, let $F(x) = \sum_{i=1}^{\infty} c_{j_i}(x)/2^{j_i+1}$ and let

$G(x) = \sum_{i=0}^{\infty} \left( \sum_{k=1}^{n_i} c_{j_i+k}(x)/2^{j_i+k+1} \right)$. Then F and G are continuous on C and, by extending F and G linearly on each interval contiguous to C, one has F and G defined and continuous on $[0,1]$. Clearly

(7) $\quad F(x) + G(x) = \varphi(x)$ on $[0,1]$.

Let $G_s(x) = -G(x) + (1-2^{n_s})\cdot\varphi(x)$ on $[0,1]$. By (7) one has

(8) $\quad G_s(x) = F(x) - 2^{n_s}\cdot\varphi(x)$ on $[0,1]$.

Clearly $G_s(0) = F(0) = 0$ and by (8) $G_s' = F'$ a.e. on $[0,1]$. Let I be a closed interval such that $I \cap C \neq \emptyset$ and $1/3^{n+1} \leq |I| < 1/3^n$, $n \in N$. Since $|I| < 1/3^n$, there exist $c_1, \ldots, c_n$ such that if $x \in C \cap I$ then $c_i(x) = c_i$, $i = 1,2,\ldots,n$. Let $A = \sum_{i=1}^{n} c_i/3^i$

and suppose that $j_{k-1} \leq n < j_k$. Let $J_1$ and $J_2$ be two closed intervals defined as follows: $J_1 = [F(A), F(A+1/3^{j_k})]$ and $J_2 = [F(A+2/3^{j_k}), F(A+3/3^{j_k})]$. Then $F(I \cap C) \subset J_1 \cup J_2$ and by (6) $|J_1| = |J_2| = \sum_{i \geq k+1} 2/2^{j_i+1} < 2/2^{j_k+1} < 2/3^{j_k} < 2/3^n < 6|I|$. Hence it follows easily that F is A(2) on C and $F \in \mathcal{F}$ on $[0,1]$.

We show now that G and $G_s$ have Lusin's property (N) on the interval $[0,1]$. Given a natural number p G(C) can be covered with $2^{n_0} \cdot 2^{n_1} \cdot \ldots \cdot 2^{n_p}$ intervals each of length at most $2/2^{j_{p+1}}$. Hence $|G(C)| = 0$, and G fulfils the property (N).

Now observe that for each $x \in C$ and $s \geq 1$,

$$-G_s(x) = (2^{n_s}) \sum_{i=0}^{s-1} \Big( \sum_{k=1}^{n_i} c_{j_i+k}(x)/2^{j_i+k+1} \Big) + (2^{n_s}-1) \sum_{i=0}^{s-1} c_{j_i}(x)/2^{j_i+1}$$

$$+ \sum_{i=0}^{\infty} \frac{(2^{n_s}-1) c_{j_{s+i}}(x) + \sum_{k=1}^{n_s} (2^{n_s-k}) c_{j_{s+i}+k}(x)}{2^{j_{s+i}+1}} +$$

$$+ (2^{n_s}) \sum_{i=0}^{\infty} \Big( \sum_{k=1}^{n_{s+i}-n_s} (c_{j_{s+i}+n_s+k}(x))/2^{j_{s+i}+n_s+k+1} \Big).$$

Since $1 + 2 + \ldots + 2^{n_s-1} = 2^{n_s}-1$, it follows that $-G_s(C)$ can be covered with $(2^{j_s-1}) \cdot \{(2^{n_s+1}-1) \cdot [(2^{n_s+1}-1) 2^{n_{s+1}-n_s}] \cdot \ldots \cdot [(2^{n_s+1}-1) 2^{n_{s+p}-n_s}]\} = [(2^{n_s+1}-1)/(2^{n_s+1})]^{p+1} \cdot (2^{j_{s+p+1}-1})$ intervals each of length at most $(2^{n_s+1})/(2^{j_{s+p+1}})$. Hence $|-G_s(C)| = |G_s(C)| = 0$, and $G_s$ has Lusin's property (N).

Remark 4. Our theorem shows that the class ACG is strictly contained in $\mathcal{F}$ .

This theorem shows also that the Banach-Zarecki theorem (see [6],pp.227) is no longer true if AC is replaced by A(N) and VB by B(N). Indeed, suppose that the Banach-Zarecki theorem remains true. Then by an argument analogous to the proof of Foran's theorem, one contradicts our theorem.

## CHAPTER V - An extension of Foran's class of functions $\mathcal{F}$ .

Definition 3. Given a real set E and a natural number N we will say that a function F is E(N) on E if for every subset S of E $|S| = 0$ and for each $\varepsilon > 0$ there exist rectangles $D_{kn} = I_k \times J_{kn}$, $n = 1,2,\dots,N$, with $\{I_k\}$ a sequence of nonoverlapping intervals, $I_k \cap S \neq \emptyset$ such that

$$B(F;S) \subset \bigcup_k \bigcup_{n=1}^{N} D_{kn} \quad \text{and} \quad \sum_k \sum_{n=1}^{N} (\operatorname{diam} D_{kn}) < \varepsilon .$$

We denote by $\mathcal{E}$ the class of all continuous functions F, defined on a closed interval I, for which there exist a sequence of sets $\{E_n\}$ and a sequence of natural numbers $\{N_n\}$ such that $I = \bigcup E_n$ and F is $E(N_n)$ on $E_n$.

Remark 5. By [2]((iii),pp.360) it follows that each function belonging to $\mathcal{E}$ also satisfies Lusin's condition (N).

Theorem 5. a) $\mathcal{E}$ is not an additive class of continuous functions,i.e., there exist two continuous functions $F_1$ and $F_2$, belonging to $\mathcal{E}$ on [0,1] such that $F_1+F_2$ does not belong to $\mathcal{E}$ . Moreover, $F_1$ and $F_2$ are differentiable a.e. on [0,1], $F_1' = -F_2'$ a.e., but $F_1+F_2$ is not identically constant.

b) $\mathcal{E}$ strictly contains the class $\mathcal{F}$ .

c) _If_ $F_1 \in \mathcal{F}$ _and if_ $F_2 \in \mathcal{E}$, _then_ $F_1+F_2 \in \mathcal{E}$.

d) $\mathcal{E}$ _is strictly contained in the class of all continuous functions, satisfying Lusin's condition_ (N).

e) _The class_ $\mathcal{E} \cap \mathcal{B}$ _is not an additive class of continuous functions, but it contains strictly the class_ $\mathcal{F}$.

_Proof_. Let $\{j_i\}$ be a strictly increasing sequence of natural numbers, such that $j_0 = 0$ and

(9) $$3^{j_i} < 2^{j_{i+1}}.$$

Let $n_i = j_{i+1} - j_i$. For each $x \in C$, we define

$$F_1(x) = \sum_{i=0}^{\infty} \left( \sum_{k=1}^{n_{2i+1}} c_{j_{2i+1}+k}(x)/2^{j_{2i+1}+k+1} \right) \quad \text{and}$$

$$F_2(x) = \sum_{i=0}^{\infty} \left( \sum_{k=1}^{n_{2i}} c_{j_{2i}+k}(x)/2^{j_{2i}+k+1} \right).$$ Then $F_1$ and $F_2$ are continuous on C and, by extending $F_1$ and $F_2$ linearly on each interval contiguous to C one has $F_1$ and $F_2$ defined and continuous on $[0,1]$. Clearly

(10) $$F_1 + F_2 = \varphi.$$

a) We show that $F_2$ is E(1) on C. (For $F_1$ the proof is similar.) Let $\varepsilon > 0$ and let p be a natural number such that

(11) $$\sqrt{2}\cdot(2/3)^{j_{2p+1}} < \varepsilon.$$

Let $I_m$, $m = 1,2,\dots,2^{j_{2p+1}}$ be the retained closed intervals from the step $j_{2p+1}$ in the Cantor ternary process, $I_m = [a_m, b_m]$. Clearly $|I_m| = 1/3^{j_{2p+1}}$. If $x \in I_m \cap C$, then $c_i(x) = c_i(b_m) = c_i(a_m)$, $i = 1,2,\dots,j_{2p+1}$; $c_i(a_m) = 0$ and $c_i(b_m) = 2$, for each $i > j_{2p+1}$. Let $J_m = [F_2(a_m), F_2(b_m)]$. Then $B(F_2;C) \subset \bigcup_m (I_m \times J_m)$

and by (9) $|J_m| = \sum_{i>p+1} \sum_{k=1}^{n_{2i}} 2/2^{j_{2i}+k+1} < 1/2^{j_{2p+2}} < 1/3^{j_{2p+1}} = |I_m|$.

By (11), $\sum \mathrm{diam}(I_m \times J_m) < \sqrt{2}\cdot |I_m| \cdot 2^{j_{2p+1}} < \varepsilon$ and $F_2 \in E(1)$ on $C$. It follows easily that $F_2 \in \mathcal{E}$ on $[0,1]$. By (10) and Remark 5, since $\varphi$ does not have Lusin's property (N), it follows that $F_1+F_2$ does not belong to $\mathcal{E}$. The second part is evident.

b) By (iii)([2],pp.360) it follows that $\mathcal{F}$ is contained in $\mathcal{E}$. Since $\mathcal{E}$ is not an additive class and since $\mathcal{F}$ is such a class, it follows that $\mathcal{E}$ strictly contains the class $\mathcal{F}$.

c) It suffices to show that if $F_1$ is $A(N_1)$ on E and $F_2$ is $E(N_2)$ on E, then $F_1+F_2$ is $E(N_1\cdot N_2)$ on E. Let S be a subset of E such that $|S| = 0$. Given $\varepsilon > 0$ let $\varepsilon_1 = \varepsilon/(2N_1+N_2)$. Let $\delta_1$ be the $\delta$ determed by $\varepsilon_1$ and the fact that $F_1$ is $A(N_1)$ on E. Let $\varepsilon_2 = \min(\varepsilon_1, \delta_1)$. There exist rectangles $D_{km} = I_k \times J_{km}$, $m = 1,2,\ldots,N_2$, where $\{I_k\}$ is a sequence of nonoverlapping intervals with $I_k \cap S \neq \emptyset$, such that

$$B(F_2;S) \subset \bigcup_k \bigcup_{m=1}^{N_2} D_{km} \quad \text{and} \quad \sum_k \sum_{m=1}^{N_2} (\mathrm{diam} D_{km}) < \varepsilon_2 .$$

Clearly $S \subset \bigcup_k I_k$ and $\sum_k |I_k| < \delta_1$. Let $J'_{kn}$, $n = 1,2,\ldots,N_1$, be intervals such that

$$B(F_1;S) \subset \bigcup_k \bigcup_{n=1}^{N_1} (I_k \times J'_{kn}) \quad \text{and} \quad \sum_k \sum_{n=1}^{N_1} |J'_{kn}| < \varepsilon_1 .$$

Let $J_{kmn} = J_{km} + J'_{kn}$ and $D_{kmn} = I_k \times J_{kmn}$. Then we obtain that

$$B(F_1+F_2;S) \subset \bigcup_{k=1} \bigcup_{m=1}^{N_2} \bigcup_{n=1}^{N_1} D_{kmn} \quad \text{and} \quad \sum_{k=1} \sum_{m=1}^{N_2} \sum_{n=1}^{N_1} (\mathrm{diam} D_{kmn}) <$$

$$< N_1\cdot N_2 \cdot \sum_k |I_k| + N_2 \cdot \sum_k \sum_{n=1}^{N_1} |J'_{kn}| + N_1 \cdot \sum_k \sum_{m=1}^{N_2} |J_{km}| \leq N_1 \cdot \varepsilon_2 + N_2 \cdot \varepsilon_1 +$$

$+ N_1 \cdot \varepsilon_2 < \varepsilon_1(2N_1+N_2) < \varepsilon$ .

d) It is well known that there is a continuous function F satisfying Lusin's condition (N) such that F(x) + x does not satisfy this property [5]. By (iii)([2],pp.360) and Remark 5, we have our result.

e) By an argument analogous to theorems 2 and 3 of [1] it follows that the above functions $F_1$ and $F_2$ belong to $\mathcal{B}$ . By part a), $F_1$ and $F_2$ belong also to $\mathcal{E}$ ; but $F_1+F_2$ does not belong to $\mathcal{E}$. Hence $\mathcal{E} \cap \mathcal{B}$ is not an additive class of functions and by (ii) ([2], pp.360), $\mathcal{E} \cap \mathcal{B}$ strictly contains the class $\mathcal{F}$ .

Theorem 6. a) If F is a continuous function belonging to $\mathcal{F}$, G is a continuous function belonging to $\mathcal{E}$ and if both are approximately differentiable a.e. on an interval I, such that $G'_{ap} = F'_{ap}$ a.e., then F-G is identically constant on I.

b) If G is a continuous function, approximately differentiable a.e. on an interval I, belonging to $\mathcal{E} - \mathcal{F}$, then $G'_{ap}$ is not integrable in the Foran sense.

Proof. a) Let H = G-F. Then $H'_{ap} = 0$ a.e. . By Theorem 5,c) and d), it follows that H satisfies Lusin's property (N). Hence by [6](pp.285-286), H is identically constant on I.

b) Suppose on the contrary that there is a continuous function G belonging to $\mathcal{E} - \mathcal{F}$ such that $G'_{ap}$ is integrable in the Foran sense. Then there is a function $H \in \mathcal{F}$ , such that $H'_{ap} = G'_{ap}$ a.e. Hence by part a) H-G is identically constant and G belongs to $\mathcal{F}$. This contradiction proves the theorem.

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