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Dan Rinne, Department of Mathematics
California State University, San Bernardino, CA 92407

On Typical Bounded Functions in the Zahorski Classes II

In [5], this author investigated properties of typical functions in terms of derived numbers. Here, we continue and extend our investigation to include typical properties stated in terms of intersections of graphs with straight lines.

All functions will be real valued with domain $I=[0,1]$. Zahorski, in [6], defined a nested hierarchy of classes of functions, $M_1 \supset \ldots \supset M_5$, and showed that M_1 is the class of Darboux Baire 1 functions ($\mathcal{DB}_1$) and M_5 is the class of approximately continuous functions. For $i=1,\ldots,5$, the class of bounded M_i functions (bM_i) is a complete metric space under the sup norm, so by a typical function we mean one belonging to a residual subset of bM_i.

The associated sets of a function, f, are sets of the form $\{x|\, f(x)>a\}$ and $\{x|\, f(x)<a\}$ for a real. We let $D_L f(x)$ and $D_R f(x)$ denote the set of derived numbers of f at x on the left and right respectively. The Lebesgue measure of a set A will be $\lambda(A)$, and $\mathcal{R}$ (resp. $\mathcal{R}^*$) will mean the set of real (resp. extended real) numbers. By $C^-(f,x)$ and $C^+(f,x)$ we mean the left and right cluster sets of f at x. Let t_f (resp. b_f) be the supremum (resp. infimum) of the set $C^-(f,x) \cup C^+(f,x)$.

In [5], we showed that the typical bM_i function has every extended real number as a derived number at every point. That is, $D_L f(x) \cup D_R f(x) = \mathcal{R}^*$ for every x in I, the obvious modifications made at the endpoints 0 and 1. In Theorem 2 of this paper, we show that $D_L f(x)=D_R f(x)=R^*$ for all x in some residual subset of I. Theorem 6 is an answer to part of Query 137 (RAE Vol. 8 No. 1) arising from Bruckner and Petruska [1].

<u>Lemma 1</u> For each i, the class of all f in bM_i such that t_f and b_f are nowhere monotonic is a residual G_δ set in bM_i.

Proof: Fix i. We show that $E=\{f \text{ in } bM_i | t_f$ is nondecreasing on some subinterval$\}$ is a first category F_σ set, the other cases having similar arguments. Let $\{I_n\}$ be the closed subintervals of I with rational endpoints and $E_n=\{f| t_f$ is nondecreasing on $I_n\}$. Then $E=\bigcup_{n=1}^{\infty} E_n$. It is easy to see that, if $f_k \to f$ uniformly, then $t_{f_k} \to t_f$ uniformly. If each f_k is in a fixed E_n, then each t_{f_k} is nondecreasing on I_n, implying t_f is nondecreasing on I_n. Thus, each E_n is closed and E is an F_σ set.

Pick f in E_n and $\varepsilon>0$. We can pick an interval $[\alpha,\beta]$ in I_n so that α and β are continuity points of f and the oscillation of f on $[\alpha,\beta]$ is less than $\varepsilon/2$. Thus, the restriction of f to $[\alpha,\beta]$, $f|[\alpha,\beta]$, is contained in the closed rectangle, S, formed by the vertical lines $x=\alpha$ and $x=\beta$ and the horizontal lines $y=f(\alpha)+\varepsilon/2$ and $y=f(\alpha)-\varepsilon/2$. We define g in bM_i so that g=f on $I-(\alpha,\beta)$ and $g|[\alpha,\beta]$ is a sawtooth function contained in S.

Then $\|f-g\| <\varepsilon$ and g is decreasing on a subinterval of (α,β). Since $t_g=g$ on (α,β), g is not in E_n. We then have E_n nowhere dense and E of first category. This completes the proof of the lemma.

Theorem 1 Let f be a Darboux function such that t_f and b_f are nowhere monotonic. Then zero is a left and right derived number on a residual subset of I.

Proof: Suppose f is Darboux and t_f and b_f are nowhere monotonic. It suffices to show that the set, A, of x in [0,1) for which the lower right Dini derivative is positive, is first category in I. We let $A_{nk}=\{x| (f(z)-f(x))/(z-x)\geq 1/k$ for $x<z<x+1/n$ and z in I$\}$. Then A is the countable union of all such A_{nk}.

Fix n and k and suppose $G=\text{int}(\text{cl}(A_{nk}))\neq\emptyset$. Pick $[\alpha,\beta]\subset G$ so that $\beta-\alpha<1/n$. We claim that b_f would then be nondecreasing on (α,β). If b_f is not nondecreasing, we can pick x<y in (α,β) with $b_f(x)= b_f(y)+r$ for some r>0. Since A_{nk} is dense in G and f is Darboux, we can then pick z in A_{nk} and w so that $x<z<w<\beta$, $f(z)>b_f(x)-r/2$, and $f(w)<b_f(y)+r/2$. Then $z<w<z+1/n$ and $(f(z)-f(w))/(z-w)<0$, contradicting our choice of z. Thus b_f is nondecreasing on (α,β), which contradicts our choice of f. We therefore have $G=\emptyset$. Thus each A_{nk} is nowhere dense and A is first category. This finishes the proof of the theorem.

An immediate result of Theorem 1 and Lemma 1 is the following.

Corollary 1 For each i, the class of all f in bM_i such that zero is a left and right derived number on a residual subset of I, is a residual subset in bM_i.

Let g be a continuous function on I. Observe that the map $\Phi:bM_i\to bM_i$ defined by $\Phi(f)=f+g$ is then a homeomorphism on bM_i. This useful observation gives us our next theorem.

Theorem 2 For each i, the class of all f in bM_i such that $D_L f(x)=D_R f(x)=R^*$ on some residual subset of I, is a residual set in bM_i.

Proof: Fix i. Let S_0 be the set of f in bM_i with 0 in $D_L f(x) \cap D_R f(x)$ for all x in V(f,0), a residual subset of I. Then S_0 is residual in bM_i by Corollary 1. For each integer n, let $S_n=\{f(x)+nx \mid f \text{ in } S_0\}$. By our observation above, each S_n is residual in bM_i, and thus so is $S=\bigcap_{n=-\infty}^{\infty} S_n$. For g in S_n, g(x)=f(x)+nx for some f in S_0, so n is in $D_L g(x) \cap D_R g(x)$ on V(g,n)=V(f,0). Thus, if f is in S, every integer is a left and right derived number on $\bigcap_{n=-\infty}^{\infty} V(f,n)$, a residual subset of I. Since f is Darboux, every extended real number is a left and right derived number on a residual subset of I. This finishes the proof of the theorem.

The remainder of this paper deals with functions in bM_i related to the graphs of straight lines, rather than questions concerning derived numbers.

Our next two theorems follow immediately from results of Garg (Corollary 3.3 and Theorem 6 of [3]) and the fact that the typical bM_i function has a dense set of discontinuities (Lemma 2 of [5]).

Theorem 3 For each i, the class of all f in bM_i so that f(x)+rx is nowhere monotonic for all r in R is a residual G_δ set in bM_i.

Theorem 4 For each i, the class of all f in bM_i so that, for every countable set $H \subset R$, there is a residual set $K \subset R$ so that the line y=mx+d intersects the graph of f in a dense in itself set for all m in H and d in K, is a residual set in bM_i.

Ceder and Pearson [2] observed that the two previous results hold for bM_1.

Theorem 5 Let f be in bM_1 such that f has every extended real number as a derived number at every x in I. Then the set of all (m,d) such that the line y=mx+d fails to intersect the graph of f in a dense in itself set is a null first category set in R^2.

Proof: We consider the empty set to be dense in itself. Suppose f is in bM_1, $D_L f(x) \cup D_R f(x) = R^*$ for all x in I, and $g \cap f$ has an isolated point, where $g(x)=mx+d$. Observe that, since f is in bM_1, $f>g$ or $f<g$ near the isolated point of $g \cap f$.

It suffices to show that the set of all (m,d) such that there is a z and a $\delta>0$ so that $f(x)>g(x)=mx+d$ on $(z-\delta, z+\delta)-\{z\}$ and $g(z)=f(z)$, is a null first category set in R^2.

We let W be the possibly larger set of all (m,d) such that there is an $x_{m,d}$ and a $\delta>0$ so that $f(x) \geq g(x)=mx+d$ on $(x_{m,d}-\delta, x_{m,d}+\delta)$ and $g(x_{m,d})=f(x_{m,d})$. We show that W is a null first category set. Let W_n be the set of (m,d) in W for which the corresponding δ is greater than or equal to 1/n. Then $W = \bigcup_{n=1}^{\infty} W_n$.

Lemma 2 Each W_n is closed.

Proof: Suppose $(m_j, d_j) \to (m,d)$ where each (m_j, d_j) is in a fixed W_n. Let $x_j = x_{m_j, d_j}$. We can then assume that $(x_j, b_f(x_j)) \to (x_0, y_0)$. Since b_f is lower semicontinuous, $b_f(x_0) \leq y_0$. For any $0<\varepsilon<1/n$, pick k so that $|x_k - x_0|$, $|m_k - m|$, and $|d_k - d|$ are each less than ε. Then, for any x in $(x_0 - 1/n + \varepsilon, x_0 + 1/n - \varepsilon)$, $|(m_k x + d_k) - (mx+d)| \leq 2\varepsilon$. Thus, $f(x) \geq mx+d-2\varepsilon$ on $(x_0 - 1/n + \varepsilon, x_0 + 1/n - \varepsilon)$, so $f(x) \geq mx+d$ on $(x_0 - 1/n, x_0 + 1/n)$. Since we also have $b_f(x_0) \leq y_0$, $b_f(x_0) = y_0$ and it is easy to see that $y_0=mx_0+d$. Thus, (m,d) is in W_n and W_n is closed. This finishes the proof of the lemma.

We now have W an F_σ set in R^2.

Lemma 3 (i) For each n and d, the set of all m such that (m,d) is in W_n is finite.

(ii) For each n and m, the set of all d such that (m,d) is in W_n is finite.

Proof: We prove part (i), a similar argument applying to part (ii). Fix n and d. If (i) is false, then $P=\{x_{m,d} \mid (m,d) \text{ is in } W_n\}$ is infinite. We can then pick $r<s$ with $x_{r,d}$ and $x_{s,d}$ in $P\cap(0,1)$ so that $|x_{r,d}-x_{s,d}|<1/n$. Then $b_f(x_{r,d})<sx_{r,d}+d$, contradicting the definition of $x_{s,d}$. Thus P is finite. This finishes the proof of the lemma.

Since $W=\bigcup_{n=1}^{\infty} W_n$, by Lemma 3 we have each d-section and m-section of W countable. Since W is measurable and each section has measure zero, $\lambda(W)=0$. Since W is an F_σ set and $\text{int}(W)=\emptyset$, W is first category. This finishes the proof of the theorem.

From Theorem 5 and the fact that the typical bM_i function has every extended real number as a derived number at every x in I (see [5]), we immediately have the following.

Corollary 2 For each i and all f in a residual subset of bM_i, the set of all (m,d) such that the line $y=mx+d$ fails to intersect the graph of f in a dense in itself set is a null first category set in R^2.

Bruckner and Petruska [1] showed that the typical function in bM_1 (also bM_5 and the class of bounded derivatives $b\Delta$) has $f^{-1}(y)$ nowhere dense and of Lebesgue measure zero for all y. Mustafa [4] has announced that this is also the case for bM_i where i=2,3,4. It turns out that the measure analogue for $cl(f^{-1}(y))$ fails in a strong way, as our next result shows.

Theorem 6 For each i, the class of all f in bM_i such that $\lambda(cl(f^{-1}(y)))>0$ for all y in some open set is a residual set in bM_i.

Proof: Fix i. Let Z be the set of functions in bM_i such that $\lambda(cl(f^{-1}(y)))>0$ for all y in some open interval. We show that Z contains a dense open set and is thus residual. Pick g in bM_i and $\varepsilon>0$. Let $\alpha<\beta$ be two continuity points of g so that the oscillation of g on $[\alpha,\beta]$ is less than $\varepsilon/4$.

Define g_1 to be equal to g on $I-[\alpha,\beta]$, $g(\alpha)$ on $[\alpha,(\alpha+\beta)/2]$, and linear on $[(\alpha+\beta)/2,\beta]$ so that g_1 is continuous on $[\alpha,\beta]$. Then g_1 is in bM_i and $||g-g_1||<\varepsilon/4$. Let u be an upper semicontinuous function in bM_5 such that $0\le u\le\varepsilon/4$ on I, u=0 on $I-(\alpha,\beta)$, u=0 on a dense subset of I, and $\lambda(T)>0$ where $T=\{x|\ u(x)=\varepsilon/4\}$, a perfect set. Zahorski [6] constructed such bM_5 functions. Then $g_2=g_1+u$ is in bM_i and we have $||g-g_2||<\varepsilon/2$. Observe that both $cl(g_2^{-1}(g(\alpha)))$ and $cl(g_2^{-1}(g(\alpha)+\varepsilon/4))$ contain T. Since g_2 is Darboux, $cl(g_2^{-1}(y))$ contains T for any $g(\alpha)\le y\le g(\alpha)+\varepsilon/4$. This follows from the fact that $b_{g_2}=g(\alpha)<g(\alpha)+\varepsilon/4=t_{g_2}$ on T.

Now suppose f is in bM_i and within $\varepsilon/16$ of g_2. Then $b_f\le g(\alpha)+\varepsilon/16<g(\alpha)+3\varepsilon/16\le t_f$ on T. If $g(\alpha)+\varepsilon/16<y<g(\alpha)+3\varepsilon/16$ and $T\not\subset cl(f^{-1}(y))$ then there is a relative subinterval, T', of T,

where $T'=T\cap(\gamma,\delta)$ and $cl(f^{-1}(y))\cap T'=\emptyset$. By our observations on b_f and t_f on T, we would then have $f^{-1}(-\infty,y)\cap(\gamma,\delta)\neq\emptyset$ and $f^{-1}(y,\infty)\cap(\gamma,\delta)\neq\emptyset$, violating the Darboux property of f. Thus, $T\subset cl(f^{-1}(y))$ for all $g(\alpha)+\varepsilon/16<y<g(\alpha)+3\varepsilon/16$. This gives an $\varepsilon/16$-neighborhood, N, of g_2 such that $N\subset Z$ and N is contained in the ε-neighborhood of g. Thus, Z contains a dense open set in bM_i. This completes the proof of the theorem.

Following the proof of Lemma 2 in [5], it is easy to show that the typical bounded derivative has a dense set of discontinuities. With this, it is easy to see that our results in Theorems 1-6 hold with bM_i replaced by $b\Delta$. The proofs of the necessary lemmas and the theorems remain unaltered.

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