Denjoy's Index and Porosity

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Section 1: Denjoy's Index

1. Introduction. This section is an exposition of Denjoy's work on a concept closely related to porosity. Porosity has been found to play an important role in differentiation theory and it is of some interest to see that over sixty years ago Denjoy had introduced a similar idea for the study of the second order symmetric derivative.

Denjoy's work appeared in three papers, [1-3, see also 14], published in 1920-21, but a more complete account appears in his book, [11, vol.II], published in 1941. All the results of this section can be found in these references, but they seem little known, even to workers in the field. Later sections will discuss how Denjoy applied his concepts, in particular to the second order symmetric derivative.

2. Some Notation. Everything in this note occurs on a bounded closed interval [a,b], and will usually be related to a non-empty perfect subset of

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[a,b], P, with a, b $\in$ P. It will easily seen that this is not essential but it makes for ease of exposition.

If $Q = \,]c, d[\, P \neq \phi$ then Q is called a portion of P. The closed interval $K = [\inf Q, \sup Q]$ is the containing segment of Q.

Remark (1) In Denjoy's work a portion means $\overline{]c, d[} \cap P (\neq \phi)$, what we will call a closed portion; it is a non-empty perfect subset of P.

If neither c, nor d, lies in P then the perfect set Q is called an isolated portion, and K an isolated segment, of P; of course, it is also a closed portion of P. In this case Q is flanked, on both sides, by contiguous intervals of P, called the isolating intervals of Q, or K.

Example (1) If $\underline{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \ldots\} = \{\varepsilon_n\}$, where $0 < \varepsilon_n < 1$, $n \geq 1$, then $C = C([a,b]; \underline{\varepsilon})$ is the symmetric perfect set constructed as follows.

Let $K_o = [a,b]$, $|K_o| = \rho_o = (b-a)$; remove from K_o the central open interval of length $\varepsilon_1\rho_o$. This leaves two symmetrically situated closed intervals, K_{11}, K_{12}, say, each of length ρ_1, where, $2\rho_1 = \rho_o(1-\varepsilon_1)$. Now remove the central open intervals of K_{11}, and K_{12}, of length $\varepsilon_2\rho_1$. Proceeding in this way we arrive at the nth stage of the construction with 2^n symmetrically situated closed intervals, K_{ni}, $1 \leq i \leq 2^n$, each of length ρ_n, where

$$2^n\rho_n = \rho_o \prod_{k=1}^{n} (1-\varepsilon_k). \tag{1}$$

If $C_n = \bigcup_{i=1}^{2^n} K_{ni}$, then $C = \bigcap_{n=1}^{\infty} C_n$; clearly, from (1),

$$|C| = \rho_o \prod_{k=1}^{\infty} (1-\varepsilon_k).$$

Certain conditions on $\underline{\varepsilon}$ are of interest.

(α) $\varepsilon_{n+1} < 2\varepsilon_n$, $n \geq 1$; this ensures that at each stage the proportion removed gets smaller.

(β) $\sum_{k=1}^{\infty} \varepsilon_n < \infty$ implies that $|C| > 0$; (in which case C is thick-in-itself).

(δ) $\varepsilon_{n+1} \leq \varepsilon_n$, $n \geq 1$; (note that this implies (α)).

(ε) $\lim_{n\to\infty} \varepsilon_n = 0$.

(μ) $\frac{1}{2} < \varepsilon_n \leq \varepsilon_{n+1}$,; (note that this implies (α)).

(ν) $\lim_{n\to\infty} \varepsilon_n = 1$.

Remarks (2) $C([0,1]; \frac{1}{3}, \frac{1}{3},\ldots\})$ is the Cantor ternary set.

(3) If $0 < \theta < \pi$, $\varepsilon_n = \frac{\theta^2}{n^2\pi^2}$, $n \geq 1$, then $|C| = (b-a)\frac{\sin\theta}{\theta}$.

So that if $0 \leq \mu < b-a$, there is a C with $|C| = \mu$.

3. The Index of a Point

Definition 1. Let $h_o > 0$, x, $x+h_o \in P$ then we define

$$\begin{aligned}\alpha_+(P;x,h_o) &= \alpha_+(x,h_o)\\ &= \inf\{\alpha;\ \exists\, h_n,\ 0 < h_{n+1} \leq h_n \leq \alpha\, h_{n+1},\ \lim_{n\to\infty} h_n = 0,\\ &\qquad x + h_n \in P,\ n \geq 0\};\end{aligned}$$

then the right index of P at x is

$$\alpha_+(P,x) = \alpha_+(x) = \alpha_+ = \inf_{h_o>0} \alpha_+(x,h_o) = \lim_{h_o\to 0} \alpha_+(x,h_o).$$

Remarks (4) Clearly $1 \leq \alpha_+ \leq \infty$, and the larger $\alpha_+(x)$, the rarer P is on the right of x.

(5) If x = b, or is the left end point of a contiguous interval of P, $\alpha_+(x) = \infty$. In general, $\alpha_+(x) = \infty$ means that $\forall x + h_n \in P$, $h_n > 0$, $n \geq 1$, $\lim_{n\to\infty} x+h_n = x$, we have $\limsup_{n\to\infty} \frac{h_{n+1}}{h_n} = \infty$.

In a similar way we can consider the left index, defining $\alpha_-(x,h_o)$, $\alpha_-(x)$, and also the (bilateral) index of x by allowing the h_n to be both positive and negative:

$$\alpha(P,x) = \alpha(x) = \alpha = \inf\{\alpha;\ \exists\, h_n,\ h_n \neq 0,\ 1 < \left|\frac{h_n}{h_{n+1}}\right| < \alpha,\ x+h_n \in P,\ n \geq 1, \text{ and } \lim_{n\to\infty} x + h_n = x\}.$$

In fact $\alpha(P,x)$ is equal to either $\alpha_+(x)$ or $\alpha_-(x)$, for the set obtained by symmetrizing P around x. Clearly,

$$\alpha \leq \tilde{\alpha} = \min(\alpha_+, \alpha_-). \tag{2}$$

Lemma 1. If the contiguous intervals of P, in [a,b], are $]a_n, b_n[$, $n \geq 1$, and we write, (using the above notation)

$$\gamma_+(P;x,h_o) = \gamma_+(x,h_o) = \sup\{\gamma;\ \frac{b_n-x}{a_n-x},\ ]a_n,b_n[\subset]\ x,x+h_o[\}$$

then $\exists\, h_n$, $0 < h_n \leq \gamma_+(x,h_o)h_{n+1}$, $\lim_{n\to\infty} h_n = 0$, $x+h_n \in P$, $n \geq 0$ and $\alpha_+(x,h_o) = \gamma_+(x,h_o)$; in particular,

$$\alpha_+(x) = \gamma_+(x) = \lim_{n\to\infty} \gamma_+(x,h_o).$$

Proof. See [11, pp. 109-111].

Remark (6) This means that $\alpha_+(x,h_o) > 1$ and is in fact an attained value of the α in Definition 1; equivalently the inf could be replaced by min there

Corollary 1. If P has unit right density at x then $\alpha_+(x) = 1$; in particular,

$\alpha(x) = 1$ a.e. on P.

Example (2). It is possible for (2) to be strict. In fact $\alpha(x)$ can be finite with $\tilde{\alpha}(x)$ infinite as the following example shows; (but see Remark (20)). Let a set on $[-1,1]$ consist of $\{0\}$ together with the closed intervals:

$$[\tfrac{1}{2}, \tfrac{1}{4}],\ [\tfrac{1}{16}, \tfrac{1}{64}],\ [\tfrac{1}{512}, \tfrac{1}{4096}], \ldots [-1, -\tfrac{1}{2}],\ [-\tfrac{1}{4}, \tfrac{1}{16}],\ [-\tfrac{1}{64}, -\tfrac{1}{512}], \ldots .$$

Then $\alpha_+(0) = \alpha_-(0) = \infty$; but on symmetrizing about 0 the set contains $[-\frac{1}{2}, \frac{1}{2}]$, so $\alpha(0) = 1$.

Lemma 1 shows the connection between Denjoy's index and porosity.

Definition 2. With the above notation put

$$p_+(P;x,h_o) = p_+(x,h_o) = \sup\{p;\ p = \frac{b_n - a_n}{h_o},\ ]a_n, b_n[\subset]x, x+h_o[\};$$

then the right porosity of P at x is

$$p_+(P,x) = p_+(x) = p_+ = \lim_{h_o \to 0} p_+(x,h_o).$$

Remark (7). Clearly, $0 \leq p_+ \leq 1$,; if $p_+(x) = 1$, then P is said to be right porous at x.

In a similar way we can define left porosity, $p_-(x,h_o)$, $p_-(x)$. Then the porosity of P at x is

$$\tilde{p}(P,x) = \tilde{p}(x) = \inf\{p_+(x),\ p_-(x)\},$$

and if $\tilde{p}(x) = 1$ then P is said to be porous at x.

Corollary 2. (a) $\alpha_{\pm} = \dfrac{1}{1 - p_{\pm}}$; (b) $\tilde{\alpha} = \dfrac{1}{1 - \tilde{p}}$.

Remark (8). It follows from Corollary 2(b), (2) and Example (2) that a set with infinite index at x is rarer than a set that is porous at x.

Theorem 1. If P is nowhere dense and $A_{\pm} = \{x;\alpha_{\pm}(x) = \infty\}$, $A = A_{+} \cap A_{-}$, then all these sets are residual subsets of P.

Proof. See [11, pp.195-196].

4. The Coefficient of Isolation.

Definition 3. (a) Let Q be an isolated portion of P, with K its containing segment, I, J its isolating intervals, then the coefficient of isolation of Q, or K, in P, is $\lambda(Q,P) = \lambda(Q) = \lambda(K,P) = \lambda(K) = \min\left\{\frac{|I|}{|K|}, \frac{|J|}{|K|}\right\}$,

(b) With the usual notation write

$$\lambda(P;x,h_o) = \lambda(x,h_o) = \sup\{\lambda;\lambda=\lambda(K),\ x \in K,\ K \subset]\,x-h_o,\ x+h_o[\}$$

then the coefficient of isolation of x in P is

$$\lambda(x,P) = \lambda(x) = \lambda = \lim_{h_o \to 0} \lambda(x,h_o).$$

Remark (9). In the definitions of $\lambda(x,h_o)$, and $\alpha(x,h_o)$, we only require that either $x+h_o$ or $x-h_o$ be in P.

(10) Clearly $0 \leq \lambda \leq \infty$, and the larger λ, the rarer P is near to x.

Example (3). In $C([a,b];\ \underline{\varepsilon})$, $\lambda(K_{ni}) = \dfrac{2\varepsilon_n}{1-\varepsilon_n}$, $1 \leq i \leq 2^n$.

The following result gives the fundamental relation between α and λ.

Theorem 2. $\operatorname{Max}\{1,\ \lambda(x,h_o)\} \leq \alpha(x,h_o) \leq 1 + 2\lambda(x,h_o)$, and the inequality on the right is strict if $\lambda(x,h_o)$ is not an attained value; in particular $\max(1,\lambda) \leq \alpha \leq 1+2\lambda$.

Proof. See [11, pp. 113-115].

Remark (11) Although $\alpha(x) = 1$ means that P is thick near to x, it is possible to have $|P| = 0$, but $\alpha = 1$ on P.

Example (4). Take the P of Definition 3 to be $C_o = C([a,b]; \underline{\varepsilon})$ with $\underline{\varepsilon}$ satisfying (α), (β) and (ε); let K, I, and J be as in Definition 3, and assume $|J| \leq |I|$. For some $n \geq 1$, $|J| = \varepsilon_n \rho_{n-1}$, J being the central removed open interval of some $K_{n-1,i}$. Hence, from (α),

$$\lambda(K) = \frac{|J|}{|K|} = \frac{\varepsilon\rho_{n-1}}{|K|} \leq \frac{\varepsilon\rho_{n-1}}{\rho_n} = \frac{2\varepsilon_n}{1-\varepsilon_n} = \lambda(K_{ni}). \tag{4}$$

From (4), and (ε), $\forall x \in P$, $\lambda(x) = 0$, and so, by (3), $\alpha(x) = 1$. Finally, from (β), $|P| = 0$.

Two simple facts concerning isolated segments are useful.

Lemma 2. (a) Let K be an isolated segment of P, c, d $\in$ P, $K \subsetneq [c,d]$, $\lambda(K) \geq \lambda$; then $(d-c) \geq (1+\lambda)|K|$.

(b) $\sum_{\lambda(K)\geq\lambda} |K| \leq (b-a)(1 + \frac{1}{\lambda})^2$.

Proof. See [11, pp.116-118].

Remark (12). Since $\lambda(K) = \infty$ only when $K = [a,b]$, (b) is equivalent to

$$\sum_{\lambda\leq\lambda(K)<\infty} |K| < (b-a)\,\frac{2\lambda+1}{\lambda^2}\,.$$

Lemma 3. If K_1, K_2 are isolated segments with $\lambda(K_1) \geq 1$, $\lambda(K_2) \geq 1$, then either $K_1 \subset K_2$, or $K_2 \subset K_1$, or $K_1 \cap K_2 \neq \emptyset$.

Proof. See [4, p.118].

Definition 4. An isolated segment K is called λ-maximal provided

(a) $K \neq [a,b]$, (b) $\lambda(K) \geq \lambda$, (c) K is not contained in another isolated segment satisfying (a) and (b); the portion of which K is the containing segment is also called λ-maximal.

Example (5). By (4) for the Cantor ternary set C we see that for all isolated segments K, of C, $\lambda(K) \leq 1$; so $K_{11} = [0, \frac{1}{3}]$, $K_{12} = [\frac{2}{3}, 1]$ are 1-maximal.

Remarks (13). If K is an isolated segment, $\lambda \leq \lambda(K) < \infty$, and if K is not λ-maximal then $\exists$ isolated segment $K_1 \supsetneq K$, $\lambda \leq \lambda(K_1) < \infty$. Hence, by Lemma 2(a), $|K_1| \geq (1 + \lambda)|K|$; This implies that after a finite number of such inclusions, $K \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots \subsetneq K_n$, K_n must be λ-maximal.

(14). If $\lambda \geq 1$, then by Lemma 3, all λ-maximal segments are disjoint.

Lemma 4. If $\lambda \geq 1$ let $\{K_\lambda\}$ be the set of λ-maximal segments, then

$$\sum |K_\lambda| \leq \frac{2(b-a)}{2+\lambda}. \qquad (5)$$

Proof. By Remark (14), $\{K_\lambda\}$ is a disjoint family ; let $\{I_\lambda\}$ be collection of intervals between the elements of this family.

Clearly $\bigcup I_\lambda \cup \bigcup K_\lambda \subset [a,b]$, and since all of these sets are disjoint, $\sum |I_\lambda| + \sum |K_\lambda| \leq b-a$.

However $|K_\lambda| \leq \frac{1}{\lambda} |I_\lambda|$, where I_λ is either of the isolating intervals of K_λ; hence $2 \sum |I_\lambda| \geq \lambda \sum |K_\lambda|$.

These two inequalities give the result.

Remark (15). Example (5) shows that (5) cannot be improved.

We now define some subsets of P that are crucial for Denjoy's applications of the index of a set.

Definition 5. For each λ, $0 \leq \lambda < \infty$, we define

(a) $J(P,\lambda) = J(\lambda) = \{x; \exists K,\ x \in K \text{ and } \lambda(K) \geq \lambda\}$;

(b) $I(P,\lambda) = I(\lambda) = P - J(\lambda)$;

(c) $\theta(P) = \theta = \bigcap_{\lambda} J(\lambda)$;

(d) $\Omega(P) = \Omega = P - \theta = \bigcup_{\lambda} I(\lambda)$.

Remarks (16). If $x \in J(\lambda)$ then x is in a λ-maximal portion of P. So $J(\lambda)$ is the union of all such portions, that are, when $\lambda \geq 1$, disjoint, (Remark (14)). In particular $J(P,\lambda)$ is open in P, and $I(P,\lambda)$ is a closed set.

(17). If x is a point of accumulation of λ-maximal portions, x $J(\lambda)$ or

(18). Remarks (16), (17) allow us to give a simple description of $J(\lambda)$, $\lambda \geq 1$. $J(\lambda)$ lies in the contiguous intervals of $I(\lambda)$, and on each such contiguous interval consists of a countable number of disjoint λ-maximal portions of P; if these are infinite in number they can only accumulate at the end points of the contiguous interval

(19). Obviously if $\lambda \leq \lambda'$ then $J(\lambda) \supset J(\lambda')$.

Lemma 5 (a) If $\lambda \geq 1$ then $|J(\lambda)| \leq \frac{2(b-a)}{2+\lambda}$.

(b) $|\theta(P)| = 0$.

(c) If $x \in I(P,\lambda)$, $x+h_o \in P \ \exists \ x+h_n \in P$, $\lim_{n\to\infty} x+h_n = x$, such that $|h_{n+1}| < |h_n| < (1 + 2\lambda)|h_{n+1}|$, $n \geq 0$; in particular $\alpha(x) \leq 1+2\lambda$.

Proof. (a) Immediate from (5).

(b) Immediate from (a).

(c) If $x \in I(\lambda)$ and $x \in K$, K an isolated segment, then $\lambda(K) < \lambda$. The result follows from Theorem 2 and Remark (6).

Corollary. $\Theta(P) = \{x; \alpha(x) = \infty\}$; $\Omega(P) = \{x; \alpha(x) < \infty\}$.

Proof. If $x \in \Omega$ then, Lemma 5(c), $\alpha(x) < \infty$.

On the other hand if $x \in \Theta$ then $\forall\lambda$, $x \in J(\lambda)$, and so $\lambda(x) = \infty$, which, by Theorem 2, means that $\alpha(x) = \infty$.

5. Denjoy's Classification of Perfect Sets.

Definition 6. If P is a non-empty perfect set then;

(a) P is of type one, $P \in D_1$, if $\exists \lambda_o$ such that if $\lambda > \lambda_o$ then $J(P,\lambda) = \phi$.

(b) P is of type two, $P \in D_2$, if $\Omega(P) = \phi$;

(c) P is of type three, $P \in D_3$, if $\Theta(P)$ and $\Omega(P)$ are both everywhere dense in P;

(d) P is of type four, $P \in D_4$, if $P \notin D_1 \cup D_2 \cup D_3$;

(e) if $P \in D_1 \cup D_2 \cup D_3$, P is said to be of definite type.

Lemma 6. If $P \in D_1$, then α is bounded on P, but not conversely.

Proof. If $P \in D_1$ then $P = I(P,\lambda)$, for $\lambda > \lambda_o$, with λ_o defined by Definition 6(a). Hence if $x \in P$, $\alpha(x) \leq 1+2\lambda_o$, by Lemma 5(c). That the converse is false can be seen by the following example. Let I_n, $n \geq 1$, denote the contiguous intervals of C_o, (in [a,b]); on each I_n place a copy of C_o on the central closed interval of length $\frac{|I_n|}{2n+1}$, P_n say. Now let $P = C_o \cup \bigcup_{n=1}^{\infty} P_n$; since $\forall x \in C_o$, $\alpha(x) = 1$, it follows that for all $x \in P$, $\alpha(x) = 1$. However $\lambda(P_n) \geq n$ and so $\bigcup_{n \geq \lambda} P_n \subset J(P,\lambda)$; in other words $\forall \lambda$, $J(P,\lambda) \neq \phi$ so $P \notin D_1$.

Let us give some examples of sets of the various types.

Example (6) Let $C_1 = C([a,b]; \underline{\varepsilon})$ where $\underline{\varepsilon}$ satisfies (δ). Then, by (4), if K is an isolated segment of C_1, $\lambda(K) \leq \frac{2\varepsilon_n}{1-\varepsilon_n} \leq \frac{2\varepsilon_1}{1-\varepsilon_1}$; taking $\lambda_o = \frac{2\varepsilon_1}{1-\varepsilon_1}$, we see that $C_1 \in D_1$. Further, from Lemma 2, or Lemma 5(c) $\forall x$, $x+h_o \in C_1 \exists x+h_n \in C_1$

$n \geq 0$, $\lim_{n\to\infty} x+h_n = x$ with $|h_{n+1}| < |h_n| < \frac{(1+3\varepsilon_1)}{1-\varepsilon_1}\cdot|h_{n+1}|$; in particular $\alpha(x) \leq \frac{1+3\varepsilon_1}{1-\varepsilon_1}$. In particular the Cantor ternary set is of type one.

Remark (19). If $P \in D_2$ then since $\Theta(P) = P$, $|P| = 0$, (Lemma 5(b)).

Example (7). Let $C_2 = C([a,b];\ \underline{\varepsilon})$ with $\underline{\varepsilon}$ satisfying (μ) and (ν). Now given any λ let n be the smallest integer for which $\frac{2\varepsilon_1}{1-\varepsilon_n} > \lambda$. Then, by (4), the K_{ni}, $1 \leq i \leq 2^n m$ are λ-maximal and so $J(P,\lambda) = C_2$; hence $C_2 \in D_2$.

Remark(20). If $P \in D_3$ then $\forall\lambda$, $J(P,\lambda)$ is everywhere dense in P, while $I(P,\lambda)$ is nowhere dense in P.

Example (8). Let $C = C([a,b];\ \underline{\varepsilon})$, with $\underline{\varepsilon}$ satisfying (γ) and (δ), and let the contiguous intervals of C on [a,b] be I_n, $n \geq 1$. On each I_n place a central closed interval J_n of length $o(|I_n|)$, as $n \to \infty$ and put $P_1 = C \cup \bigcup_{i=1}^{\infty} J_n$. Replace, in P_1, each J_n by a copy of P_1, to obtain a set P_2, $P_2 \subset P_1$. The set P_2 consists, as did P_1, of a nowhere dense, thick-in-itself, perfect set, together with a set of central closed intervals on certain of its contiguous intervals. Replace, in P_2, each of these central closed intervals by a copy of P_1, to obtain P_3, $P_3 \subset P_2$. Repeating this define $C_3 = \bigcap_{n=1}^{\infty} P_n$. Then $C_3 \in D_3$ and is thick-in-itself, see [11, pp.123-125].

(9) On [a,b] place the points a_n with $a < a_n < a_{n+1} < b$, $n \in Z$, and $\lim_{n\to-\infty} a_n = a$, $\lim_{n\to\infty} a_n = b$. If $n \equiv i(\text{mod } 3)$, place on $[a_n, a_{n+1}]$ a copy of C_i, $1 \leq i < 3$, P_n say; if then $C_4 = \{a\} \cup \{b\} \cup \bigcup_{n=-\infty}^{\infty} P_n$, $C_4 \in D_4$.

Denjoy proves two theorems that elucidate the structure of certain perfect sets.

Theorem 3. If $P = \Omega(P)$ and $x \in P$ define

$$\alpha'(x) = \max\{\alpha', \forall h_o, x+h_o \in P \ \exists \ x+h_n \in P, \lim_{n\to\infty} x+h_n = x \text{ and } |h_{n+1}| < |h_n| \leq \alpha' |h_{n+1}|\};$$

then either α' is bounded on P or $\exists$ a closed nowhere dense subset of P, H, such that if Q is closed portion of P, $Q \cap H = \phi$, then α' is bounded on Q.

Corollary 1. If $P = \Omega(P)$ and $P \notin D_1$ then $\exists$ a closed nowhere dense set H such that if Q is an isolated portion of P, $Q \cap H = \phi$, then $Q \in D_1$.

Proof. See [11, pp.148-149].

Corollary 2. If α is a finite on P then at some point of P both α_+ and α_- are finite.

Proof. From Corollary 1 it is sufficient to prove this for sets $P \in D_1$.

Let us suppose then that for some λ, $P = I(P,\lambda)$ and further assume that $\forall x \in P$ either $\alpha_+(x)$ or $\alpha_-(x)$ is infinite.

Choose $x \in P$ isolated on the left, by an interval I say, and assume that arbitrarily near to x, on the right, all points of P of the second type, y say, have $\alpha_+(y) = \infty$. Choose one x+h, say, so that $\frac{|I|}{2h} > \lambda$. If K is an isolated segment [x,x'] in [x,x+2h], with J its right isolating segment then

$$\lambda > \lambda(K) = \min\left\{ \frac{|I|}{|K|}, \frac{|J|}{|K|} \right\} = \frac{|J|}{|K|}.$$

Consider a sequence of contiguous intervals of P in [x+h, x+2h] converging to x+h; since $\alpha_+(x+h) = \infty$ there is one, $J = [x+h_1, x+h_o]$ say, with

$\frac{h_o}{h_1} > \lambda + 1$; but then if we take $K = [x, x+h_1]$ we have that $\lambda > \lambda(K) = \frac{|J|}{|K|} = \frac{h_o - h_1}{h_1} > \lambda$, a contradiction.

If all the points of second type near to x on the right have $\alpha_+(y) \leq \infty$ they must have $\alpha_-(y) = \infty$. Then we can choose an x', isolated on the right, with all points y arbitrarily near to it, of the second type having $\alpha_-(y) = \infty$, and we can proceed as above.

Remark (20). In particular this shows that while a point can be a point of porosity and have a finite index, this cannot happen at all points of a set.

Theorem 4. If $P \varepsilon D_4$ and

$H = \{x;$ x is not in a closed portion of definite type$\}$

then H is a closed nowhere dense subset of P.

Proof. Suppose P contains a portion Q, no closed sub-portion of which is of definite type. Then clearly Q is nowhere dense since intervals are of type one. If then $\tilde{Q}$ is an isolated portion of Q and if $\forall x \varepsilon \tilde{Q}$ $\alpha(P,x)$ is finite, then $\alpha(Q,x)$ is finite, and so by Theorem 3 either $\tilde{Q} \varepsilon D_1$ or $\tilde{Q}$ contains an isolated portion of type one.

If $\forall x \varepsilon \tilde{Q}$, $\alpha(P,x) = \infty$ then $\tilde{Q} \varepsilon D_2$. So finally, if Q contains no portions of definite type then $\Omega(P)$ and $\Theta(P)$ meet every portion of Q and so $Q \varepsilon D_3$.

Remark (21). If then P is nowhere dense, $P \varepsilon D_4$ and $x \notin H$ then x belongs to an isolated portion of definite type. However if Q is an isolated portion of P, $Q \cap H = \phi$ it need not be of definite type; but it is the finite union of disjoint isolated portions of definite type.

(22). A contiguous interval of H can contain infinitely many isolated portions of definite type; any two are either neighbours of different type, or one separated by at least one portion of different type.

Section 2: Applications of Denjoy's Index - I.

1. Introduction. In this section a survey is given of Denjoy's use of his index to obtain deep properties of the second symmetric derivative. While some of the results have been proved, and extended, by other methods, (see the papers of Marcinkiewicz and Zygmund, Bullen and Mukhopadhyay), Denjoy's techniques are more elementary, very closely connected to interesting properties of the real line, and give some very precise results not obtained by the other methods. His arguments being based on his concept of index should be of interest to workers in differentiation, and deserve to be more widely known. Denjoy himself considered this work to be second only to his fundamental research on primitives, (see the article by Cartan in Denjoy [13]), but his papers are too short, and his book, (Denjoy [11]), contains such a wealth of detail, and is so general that the main line of reasoning is sometimes obscured.

2. Preliminaries. If $F : [a,b] \to R$ is continuous and such that $D_s^2F = f$ exists, (finite), at all points of $]a,b[$, where

$$D_s^2F(x) = \lim_{h\to 0} \frac{\Delta_s^2F(x,h)}{h^2},$$

$$\Delta_s^2F(x,h) = F(x+h) + F(x-h) - 2F(x),$$

then we will say f is N_s^2-integrable, $f \in N_s^2$, with F, unique up to an affine function, an N_s^2-primitive of f, $F = N_s^2 - \int f$. The unique such primitive that is zero at a and at b is

$$\tilde{F}(x) = F(x) - F(a) + \frac{x-a}{b-a}(F(b) - F(a)); \tag{1}$$

and we will write

$$N_s^2 - \int_{(a,b)}^{x} f = -\tilde{F}(x) = (x-a)(b-x)V_2(F;a,x,b), \tag{2}$$

where

$$V_2(F;a,x,b) = \frac{F(a)}{(x-a)(b-a)} - \frac{F(x)}{(b-x)(x-a)} + \frac{F(b)}{(b-a)(b-x)}, \tag{3}$$

$$= \frac{1}{b-a}\left\{\frac{F(b)-F(x)}{b-x} - \frac{F(x)-F(a)}{x-a}\right\},$$

is the second divided difference of F at a, x, b.

If also $f \in L$ then an N_s^2-primitive is

$$F(x) = \int_a^x \int_a^u f(t)dtdu$$

and

$$N_s^2 - \int_{(a,b)}^{x} f = \frac{1}{b-a}\left((b-x)\int_a^x (t-a)f(t)dt + (x-a)\int_x^b (b-t)f(t)dt\right). \tag{4}$$

(Integral signs not prefixed will always denote Lebesgue integrals, unless the context makes another meaning obvious.)

Remark (1). $D_s^2F(x)$ is called the second symmetric derivative of F at x; the upper and lower second symmetric derivatives of F at x are easily defined, and are written $\overline{D}_s^2F(x)$, $\underline{D}_s^2F(x)$ respectively.

The object of this paper is obtain differential properties of N_s^2-primitives, or more generally of $F : [a,b] \to R$ that are continuous and satisfy

$$-\infty < \underline{D}_s^2F \le \overline{D}_s^2F < \infty. \tag{5}$$

If then f is a function such at $\underline{D}_s^2 F(x) \leq f(x) \leq \overline{D}_s^2 F(x)$ let us say f is $N_{s,g}^2$-integrable, and that F is a $N_{s,g}^2$-primitive.

Remarks (2). If F is a $N_{s,g}^2$-primitive it is smooth. If we require F to be a smooth then we need only require (5) to hold n.e.

(3). Denjoy obtains the results in such a precise form that given a $f \in N_{s,g}^2$ it is possible to calculate $N_s^2\text{-}\int_{(a,b)}^{x} f$. This topic will be taken up in section 3.

By familiar arguments, see for instance Saks, pp. 238-240, [a,b] can be expressed as a countable union of closed sets $E = E(A) = E(A,\varepsilon)$, $A > 0$, $\varepsilon > 0$, for which

$$\forall\, x \in E,\ 0 \leq h \leq \varepsilon,\ |\Delta^2 F(x,h)| \leq Ah^2. \tag{A}$$

So we first consider what can be deduced about an F defined on [a,b] that satisfies (A) on a set E that consists of one, two, three, four, any finite number of points, is an interval, is perfect.

3. The Process of Successive Symmetries [11, III, pp.231-244]

If in (A), $A = 0$ and $\Theta_1 \in E$ then $P_1(\Theta_1, F(\Theta_1))$ is a centre for the graph of F; that is if (u,v) lies on the graph so does (s,t) where $\Theta_1 = \frac{1}{2}(u+s)$, $F(\Theta_1) = \frac{1}{2}(v+t)$. If also $\Theta_2 \in E$ then P_1 and $P_2(\Theta_2, F(\Theta_2))$ are both centres and the graph of F is periodic about the line through P_1P_2, with period $2|P_1P_2|$, oscillating equally on both sides of this line; (if further $\Theta_3 \in E$ then P_1, P_2 and $P_3(\Theta_3, F(\Theta_3))$ are collinear). Suppose $u \neq \Theta_1, \Theta_2$ then from $M(u,F(u))$ we can construct a doubly infinite sequence of points on the graph by taking images alternatively through P_1 and P_2, starting with either P_1, or P_2. Thus M_1 is the image of M through P_1, M_2 the image of M_1 through P_2, etc; while M_{-1} is the image of M through P_2, M_{-2} the image of M_{-1} through P_1 etc. We will call this the process of successive symmetries (SS). Let us write $\Theta_2 = \Theta_1 + k$, and assume $k > 0$, $u = u_o = \Theta_1 + h$; if, for $m \in Z$, u_m is the first coordinate of M_m then:

$$u_{2m} = u + 2mk,$$

$$u_{2m-1} = 2\Theta_2 - u_{2m} = 2\Theta_2 - u - 2mk;$$

or (SS)

$$u_{2m}-\Theta_1 = \Theta_1 - u_{2m+1} = h + 2mk,$$

$$u_{2m}-\Theta_2 = -(u_{2m}-\Theta_2) = - h - (2m-1)k.$$

(In (11, p.232) there is a diagram that illustrates (SS).)

If $A > 0$ the above remarks are only approximately true, but (A) can be used to estimate the error in these approximations.

Remark (4). Throughout δ, δ', δ_1 etc. will always denote numbers, not necessarily the same, between -1 and 1.

Lemma 1. (a) If (A) holds at Θ, F finite, and if $u + u_1 = 2\Theta$ then

$$F(u_1) = - F(u) + 2F(\Theta) + \delta A(u-u_1)^2$$

(b) If (A) holds at Θ_1, Θ_2 with $\Theta_1 < \Theta_2$, F finite and if u_m, $m \in Z$, is defined by (SS), and D is the largest of the distances of u and u_{2m} from $\{\Theta_1,\Theta_2\}$ then

$$F(u_{2m}) = F(u) + 2m\{F(\Theta_2) - F(\Theta_1) + \delta AD \}, \qquad (6)$$

or

$$\frac{F(u_{2m})-F(u)}{u_{2m}-u} = \frac{F(\Theta_2)-F(\Theta_1)}{\Theta_2-\Theta_1} + \frac{\delta A D^2}{\Theta_2-\Theta_1} \qquad (6')$$

(c) If (A) holds at $\Theta_1, \Theta_2, \Theta_3$, F finite measurable, (or bounded), $|\Theta_2 - \Theta_1| \le |\Theta_3 - \Theta_1|$, $D = \text{diameter}\{\Theta_1, \Theta_2, \Theta_3\}$, then

$$\frac{F(\Theta_3)-F(\Theta_1)}{\Theta_3 - \Theta_1} - \frac{F(\Theta_2)-F(\Theta_1)}{\Theta_2 - \Theta_1} = \frac{3\delta A D^2}{\Theta_2-\Theta_1} \qquad (7)$$

(d) If (Å) holds $\Theta_1, \Theta_2, \Theta_3, \Theta_4$, F finite measurable, (or bounded), $|\Theta_2-\Theta_1| \le |\Theta_3-\Theta_4|$, $|\Theta_4-\Theta_1| = \max|\Theta_i-\Theta_j|$, then

$$\frac{F(\Theta_4)-F(\Theta_3)}{\Theta_4-\Theta_3} - \frac{F(\Theta_2)-F(\Theta_1)}{\Theta_2-\Theta_1} = 3\delta A \frac{(\Theta_4-\Theta_1)^2}{\Theta_2-\Theta_1}, \qquad (8)$$

or

$$F(\Theta_2)-F(\Theta_1) = (\Theta_2-\Theta_1)\frac{F(\Theta_4) - F(\Theta_3)}{\Theta_4 - \Theta_3} + \delta A(\Theta_4-\Theta_1)^2 \; 1 + \frac{2(\Theta_2-\Theta_1)}{\Theta_4-\Theta_3} \; . \qquad (8')$$

Proof. See [11, III, pp.235-244].

Remarks (5) While (c) can be proved directly it is also a consequence of (d), putting $\Theta_3 = \Theta_1$, or $\Theta_3 = \Theta_2$.

(6) If $2|\Theta_2-\Theta_1| < |\Theta_3-\Theta_4|$ the coefficient 3 in (7), and (8), can be improved to 2.

4. Variational Properties. If now F is finite measurable, (or bounded), and satisfies (A) at n + 1 points u_i, $0 \le i \le n$, ordered as $c = u_o < u_1, \ldots < u_n = d$ then from (8') and for $1 \le i \le n$,

$$F(u_i) - F(u_{i-1}) = \left(\frac{u_i - u_{i-1}}{d - c}\right)\{F(d) - F(c)\} + 3\delta A(d-c)^2,$$

which on adding gives

$$\sum_{i=1}^{n} |F(u_i) - F(u_{i-1})| < |F(d) - F(c)| + 3nA(d-c)^2. \tag{9}$$

However, when n > 3, (9) can be improved.

Lemma 2. If F is finite measurable, (or bounded), and satisfies

(A) at u_k, $0 \le k \le n$, $c = u_o < \ldots < u_n = d$, then

$$\sum_{k=1}^{n} |F(u_i) - F(u_{n-1})| < |F(d) - F(c)| + 18A(d-c)^2. \tag{10}$$

Proof. If σ is the partition of [c,d] given above we will arrive at σ by a finite number of intermediary subdivisions, $\sigma_o,\ldots,\sigma_m = \sigma$, by adding the points in a particular way; we will renumber $u_o,\ldots,u_n$ in the order in which they are used, calling them $v_o,\ldots,v_n$.

(i) σ_o is just $v_o = c$, $v_1 = d$; let $S_o = |F(v_1) - F(v_o)|$.

(ii) σ_1 is formed by adding, to σ_o, the two consecutive points of σ that straddle $\frac{c+d}{2}$, or if $\frac{c+d}{2}$ is a point of σ, by just adding this point to σ_o. Thus σ_o is either v_o,v_2,v_3,v_1, or v_o,v_2,v_1; in either case $K_o = [c,d]$ can be considered as the union of two closed intervals $K_{11} = [v_o,v_2]$, $K_{12} = [v_3,v_1]$ and a, possibly empty, open interval I_{11}, and $|K_{1i}| \le \frac{b-a}{2}$, $i = 1,2$. Applying (9) to σ_1 we get

$$S_1 = |F(v_2)-F(v_o)| + |F(v_3)-F(v_2)| + |F(v_1)-F(v_3)| < |F(a)- F(c)| + 9A(d-c)^2$$

(iii) Now add points to K_{11}, K_{12} in the same way as points were added to K_o; this gives σ_2. Then [c,d] is the union of four closed intervals, K_{2i}, $|K_{2i}| \le \frac{d-c}{4}$, $1 \le i \le 4$, and at most three open intervals I_1, I_{2i}, $1 \le i \le 2$. Applying (9) to both K_{11} and K_{12} we get the sum over this partition, S_2, satisfies

$$S_2 < S_1 + 9A(|K_{11}|^2 + |K_{12}|^2) \leq |F(d) - F(c)| + \frac{27}{2} A(d-c)^2.$$

Continuing in this way we find that at the qth stage, [c,d] is the union of 2^q closed intervals K_{qi}, $|K_{qi}| \leq \frac{b-a}{2^q}$, $1 \leq i \leq 2^q$, and at most 2^q - 1-open intervals I_{pi}, $1 \leq p \leq q$, $1 \leq q$, $1 \leq i \leq 2^{p-1}$. Further the sum over this partition is S_q and

$$S_q < |F(d)-F(c)| + 9A \sum_{p=0}^{q-1} \sum_{j=1}^{2^p} |K_{pj}|^2 \leq |F(d)-F(c)| + 18A(d-c)^2(1 - \frac{1}{2^q}) \quad (11)$$

Since for some q, $\sigma_q = \sigma$ this completes the proof.

Remarks (7). If F, finite measurable, or bounded, satisfies (A) at all points of a nowhere dense perfect set P with extremities c, d then (11) holds for all q since we can use the contiguous intervals to perform the above construction. Let I_1 be the contiguous interval whose closure contains $\frac{c+d}{2}$, or take $I_1 = \phi$ it $\frac{c+d}{2}$ is a point of P of the second type : in either case we can write [c,d] as the union of two closed intervals K_{1i}, $|K_{1i}| \leq \frac{d-c}{2}$, containing P, and a, possibly empty, open interval I_1; this procedure can be repeated in definitely.

(8) Clearly from Lemma 2 if F, finite measurable, satisfies (A) at all points of an interval [c,d] then $F \in BV$ and its variation is bounded by the right hand side of (10).

However more is true.

Theorem 1. If F is continuous and satisfies (A) at all points of a perfect set P then (a) $F \in B\,V\,(P)$, (b) if $|P| = 0$, $F \in AC(P)$.

Proof. It suffices to consider P a nowhere dense perfect set.

Let $a = b_o < a_1 < b_1 < \ldots < b_n < a_{n+1} = b$, where $[a_k, b_k]$, $1 \le k \le n$, are the first n of the closed contiguous intervals of P enumerated in any order. Then, by (10)

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| + \sum_{k=1}^{n+1} |F(a_k) - F(b_{k-1})|$$
$$< |F(b) - F(a)| + 18A(b-a)^2.$$

This suffices to prove (a). Further if

$$VT(F,P) = \lim \sum_{k=1}^{n+1} |F(a_k) - F(b_{k-1})|,$$

(Denjoy's total variation of F on P, see Denjoy [11, §45]), then

$$VT(F,P) \le |F(b) - F(a)| + 18A(b-a)^2.$$

If we apply this inequality to

$$F_1(x) = F(x) - F(a) - \frac{x-a}{b-a}\{F(b) - F(a)\},$$

we get

$$VT(F_1,P) \le 18A(b-a)^2;$$

and of course for any c, d, $a \le c < d \le b$

$$|F(d) - F(c)| \le |F_1(d) - F_1(c)| + \frac{d-c}{b-a} |F(b) - F(a)|$$

Hence

$$VT(F,P) \le VT(F_1,P) + \rho|F(b) - F(a)|$$
$$\le \rho|F(b) - F(a)| + 18A(b-a)^2,$$

where $\rho = \lim_{n\to\infty} \frac{1}{b-a} \sum_{k=1}^{n+1} (a_k - b_{k-1})$ = average density of P on [a,b]. In particular if $|P| = 0$,

$$VT(F,P) \le 18A(b-a)^2.$$

Clearly this inequality can be applied with a, b, P replaced by any c, $d \in P$, $P \cap [c,d]$ respectively. Hence

$$VT(F,P) = \sum_{k=1}^{n+1} VT(F, P \cap [b_{k-1}, a_k])$$

$$\leq 18A \sum_{k=1}^{n+1} (b_{k-1} - a_k)^2;$$

letting $n \to \infty$ implies $VT(F,P) = 0$, which completes the proof of (b).

Corollary 1. If F is a $N^2_{s,g}$-primitive then F ACG; in particular F'_{ap} exists a.e. and is D-integrable.

Proof. This is an immediate consequence of Theorem 1(b) and an important result of Denjoy; see Lebesgue, p.234.

5. Differential Properties

Remarks (9). If $\alpha(P,x) = \alpha(x) < \infty$, (see section 1), then

$\alpha' > \alpha \ \exists\ x + h_n \in P$, with $|h_{n+1}| < |h_n| < \alpha'|h_{n+1}|$, $n \geq 1$, and $\lim_{n\to\infty} h_n = 0$.

It is possible, by choosing a suitable subsequence if necessary, to assume that $\exists\, x + h_n \in P$, such that $2|h_{n+1}| \leq |h_n| < 2\alpha'|h_{n+1}|$, $n \geq 1$, $\lim_{n\to\infty} h_n = 0$.

(10) If $2|k_{n+1}| \leq |k_n| < \beta|k_{n+1}|$, $n \geq 0$ then

$$\omega_n = \sum_{i \geq n} \frac{k_i^2}{|k_{i+1}|} < \beta|k_{n-1}|, \ n \geq 1.$$

Lemma 3. Let F be continuous at x and satisfy (A) at x and at $x + k_n$, where $2|k_{n+1}| \leq |k_n| < \beta|k_{n+1}|$, $n \geq 1$, and $\lim_{n\to\infty} k_n = 0$, then $F'(x)$ exists.

Proof. We may assume, by omitting some points if necessary, that $2|k_1| < \varepsilon$, (ε in (A)); let $Q_n = \dfrac{F(x+k_n)-F(x)}{k_n}$ and we first show that $\lim_{n\to\infty} Q_n$ exists.

Apply (7), (with Remark (6)), to $\Theta_1 = x$, $\Theta_2 = x+k_{n+1}$, $\Theta_3 = x+k_n$, when $|D| \leq |k_n| + |k_{n+1}| < 2|k_n|$;

$$Q_{n+1} = Q_n + 8\delta_n A \frac{k_n^2}{|k_{n+1}|},$$

and so

$$Q_{n+p} = Q_n + 8A \sum_{i=n}^{n+p-1} \delta_i \frac{k_i^2}{|k_{i+1}|}.$$

So, by Remark (10), $\lim_{n\to\infty} Q_n$ exists, Q say, and with the notation of Remark (10).

$$Q_n = Q + 8A\delta_n \omega_n. \tag{12}$$

We now want to estimate $\frac{F(x+h)-F(x)}{h} - Q$, and first assume $2|k_1| < h < \varepsilon$. Apply (SS) with $u_o = x+h$ and Θ_1, Θ_2 as $x, x+k_1$ in some order; ($\Theta_1 = x$, $\Theta_2 = x+k_1$ if $k_1 > 0$, the other way if $k_1 < 0$). Write $u_1 = x+h_1 = u_{-2m_1}$; where $m_1 > 0$ is given by $m_1 < \frac{h}{2|k_1|} \leq m_1 + 1$.

Suppose now $u_1 = x+h_1, \ldots, u_{j-1} = x+h_{j-1}$ have been obtained, together with the positive integers $m_1, \ldots, m_{j-1}$; apply (SS) with $u_o = u_{j-1}$ and Θ_1, Θ_2 as $x, x+k_j$ in some order; write $u_j = x+h_j = u_{-2m_j}$ where $m_j > 0$ is given by $m_j < \frac{h_{j-1}}{2|k_j|} \leq m_j + 1$. Then we can easily verify the following:

$$\begin{aligned} &0 < h_j \leq 2|k_j|, \quad m_j < \frac{h_{j-1}}{2|k_j|} \leq \left|\frac{k_{j-1}}{k_j}\right|, \quad j \geq 1; \\ &h = \sum_{j\geq 1} 2m_j|k_j|, \quad h_n = \sum_{j\geq n+1} 2m_j|k_j|, \quad n \geq 1. \end{aligned} \tag{13}$$

Now, by (6), $j \geq 2$,

$$F(x+h) - F(x+h_1) = 2m_1|k_1|(Q_1 + \delta_1 A \frac{h^2}{|k_1|}),$$

$$F(x+h_{j-1}) - F(x+h_j) = 2m_j|h_j|(Q_j + \delta_j A \frac{h_{j-1}^2}{|k_j|} ;$$

or using (12) and (13),

$$F(x+h) - F(x+h_1) = 2m_1|k_1|(Q + 8\delta_1 A\omega_1) + \delta_j' A \frac{h^3}{|k_1|}$$

$$F(x+h_{j-1}) - F(x+h_j) = 2m|k_j|(Q + 8\delta_j A\omega_j + 4\delta_j' A \frac{k_{j-1}^2}{k_j})$$

$$= 2m_j|k_j|(Q + 4\delta_j A\omega_j + 4\delta_j' A\omega_{j-1})$$

$$= 2m_j|k_j|(Q + 8\delta A\omega_1).$$

Adding these results, noting that $\lim_{j\to\infty} F(x+h_j) = F(x)$, we get, by (13), that

$$F(x+h) - F(x) = hQ + 8\delta Ah\omega_1 + \delta' A \frac{h^3}{|k_1|} \tag{14}$$

Hence if we assume that $h < \beta|k_1| < \varepsilon$, by Remark (10),

$$F(x+h) - F(x) = hQ + 16\delta\, A\beta h^2; \tag{15}$$

and by a similar argument (14) and (15) hold if $2|k_1| < |h| < \beta|k_1| < \varepsilon$.

If we now assume that $2|k_q| < |h| \leq 2|k_{q-1}|$ we can also prove that

$$F(x+h) - F(x) = hQ + 16\delta Ah\omega_q + \delta' A \frac{h^3}{|k_q|} ; \tag{14-q}$$

and if in addition $2 < |\frac{h}{h_q}| < \beta$ then (15) again holds.

This proves that $F'(x)$ exists, with value Q.

Remark (11). Let P be a perfect set, $P \subset E(A)$, and suppose $\exists\ \alpha$ such that $\forall\ x$, $x+k_1 \in P\ \exists\ x+k_n \in P$ with $k_n \to 0$ and $|k_{n+1}| < |k_n| < \alpha|k_{n+1}|$, then, by Remark (9) we can apply (15) to any two points of P with $\beta = 2\alpha$. Hence if x, $x+h \in P$

$$F(x+h) - F(x) = hF'(x) + 32\delta A\alpha h^2,$$
$$= hF'(x+h) + 32\delta' A\alpha h^2;$$

which gives that $|F'(x+h) - F'(x)| < 64A\alpha|h|$: or F' is continuous on P.

(12). The above remarks applies in particular to any $P \in D_1$, and to $I(\lambda) = I(E,\lambda)$, when $\alpha = 2\lambda + 1$, (Lemma 5(c) of Section 1). In the latter case, if D is the diameter of $I(\lambda)$, then $\forall x \in I(\lambda)$, $|h| < (2\lambda + 1)D$

$$F(x+h) - F(x) = hF'(x) + 32\delta A(2\lambda + 1)h^2; \qquad (15\text{-}\lambda)$$

and if x, y are on $I(\lambda)$

$$F(y) - F(x) = (y-x)F'(x) + 32\delta A(2\lambda + 1)(y-x)^2,$$
$$= (y-x)F'(y) + 32\delta' A(2\lambda + 1)(y-x)^2, \qquad (16)$$

and so

$$F'(y) - F'(x) = 64\delta A(2\lambda + 1)(y-x). \qquad (17)$$

Corollary 2. Let P be a perfect subset of $E(A)$ for which $\Theta(P)$ is nowhere dense. Then, except at the points of nowhere dense subset of P, F' exists and is continuous relative to P.

Proof. By theorem 2 of section 1 this is an immediate consequence of Remark (11).

Corollary 3. On $I(\lambda)$, F' is continuous and satisfies a Lipschitz condition; further $(F')'_{I(\lambda)}$ exists a.e.

Proof. The first part is an immediate consequence of Remarks (11) and (12).

Let c, d, $c < d$, be the extremities of $I(\lambda)$ and define Φ to be F' on $I(\lambda)$, linear on the contiguous intervals, and continuous. Then Φ satisfies the same Lipschitz condition as F' and so Φ' exists a.e. on $[c,d]$. Clearly at all non-isolated points of $I(\lambda)$ where $\Phi'(x)$ exists we have

$$\Phi'(x) = \lim_{\substack{y\to x\\ y\in I(\lambda)}} \frac{F'(y)-F'(x)}{y-x} = (F')'_{I(\lambda)}(x).$$

Remark (13). In particular $(F')'_{ap}$ exists a.e. on $I(\lambda)$, and so, by Lemma 5(b) of section 1, a.e. on $E(A)$.

(14). Let us write

$$\begin{aligned} \phi(x) &= (F')'_{I(\lambda)}(x), \text{ whenever this exists,} \\ &= 0, \text{ elsewhere on } [c,d] \end{aligned} \tag{18}$$

Further denote by $]c_n,d_n[$, $n \geq 1$, the contiguous intervals of $I(\lambda)$, on $[c,d]$; and for $n \geq 1$,

$$\begin{aligned} s_n &= \frac{F(d_n) - F(c_n)}{d_n - c_n} - \frac{1}{2}\{F'(c_n) + F'(d_n)\} \\ t_n &= F'(d_n) - F_p(c_n). \end{aligned} \tag{19}$$

From (16),

$$|s_n| < 32A(2\lambda+1)(d_n-c_n);$$

from (17) (20)

$$|t_n| < 64A(2\lambda+1)(d_n-c_n).$$

Corollary 4. Let

$$H = \{x;\ x \in I(\lambda),\ (F')'_{I(\lambda)}(x) \text{ exists, and } \alpha_+(x) = \alpha_-(x) = 1\},$$

then if $x \in H$, $F_{(2)}(x)$ exists; precisely

$$F_{(2)}(x) = \phi(x),\ F(x+h) - F(x) = hF'(x) + \frac{h^2}{2}\phi(x) + o(h^2).$$

Proof. We introduce two auxiliary functions; G, being the simplest function agreeing with F on $I(\lambda)$ and having the same derivatives as F at c_n, d_n, $n \geq 1$; and $M = F - G$. Simple calculations show that if $c_n < x < d_n$, then

$$G'(x) = F'(c_n) + \frac{x-c_n}{d_n-c_n}(F'(d_n) - F'(c_n)) + \mu_n \frac{(x-c_n)(d_n-x)}{d_n - c_n};$$

$$G(x) = F(c_n)+(x-c_n)F'(c_n) + \frac{(x-c_n)^2}{2(d_n-c_n)}(F'(d_n)-F'(c_n))$$

$$+ \mu_n \frac{(x-c_n)^2(d_n-x)}{2(d_n-c_n)} + \frac{(x-c_n)^3}{6(d_n-c_n)}\};$$

where

$$\mu_n = \frac{6s_n}{d_n-c_n};$$

$$M(x) = F(x)-F(c_n)-(x-c_n)-(x-c_n)F'(c_n)-(x-c_n)^2\{\frac{F'(d_n)-F'(c_n)}{2(d_n-c_n)} + \mu_n \frac{d_n-x}{2(d_n-c_n)}$$

$$+ \mu_n \frac{x-c_n}{6(d_n-c_n)}\}.$$

From (20), we have $|\mu_n| < 192A(2\lambda+1)$, and so from the above definitions, and (17), if $c_n < x < d_n$, $\left|\frac{G'(x)-F'(c_n)}{x - c_n}\right| < 256A(2\lambda+1)$, and similarly with c_n replaced by d_n. Further from (20), and (16), if $c_n \le x \le d_n$

$|M(x)| < 160A(2\lambda+1)(x-c_n)^2$, and similarly with c_n replaced by d_n.

Suppose now $\Theta \in H$ and consider

$$\frac{M(x)-M(\Theta)}{\Theta - x} = \frac{M(x)}{\Theta-x}, \quad c_n < x < d_n$$

$$= 0, \text{ otherwise.}$$

From hypothesis, $\alpha_{\pm}(x) = 1$, it follows that both $\frac{x-c_n}{x-\Theta}$ and $\frac{d_n-x}{x-\Theta}$ tend to zero as $x \to \Theta$, (see section 1). Hence $M'(x) = 0$ and $M_{(2)}(x) = 0$.

We now show that $G''(x) = \phi(x)$, which will complete the proof. From the above definitions and estimates, if x, y are any two points of $[c,d]$

$$|G'(x) - G'(y)| < 256A(2\lambda+1)|x-y|. \tag{21}$$

Suppose then $\Theta \in H$ and $x \in I(\lambda)$ then

$$\frac{G'(\Theta) - G'(x)}{\Theta - x} = \frac{F'(\Theta) - F'(x)}{\Theta - x}$$

which tends to $\phi(\Theta)$ as $x \to \Theta$. If however $c_n < x < d_n$, then

$$\frac{G'(\Theta) - G'(x)}{\Theta - x} = \frac{G'(x) - F(c_n)}{x - c_n} \cdot \frac{x - c_n}{\Theta - x} + \frac{F'(c_n) - F'(\Theta)}{c_n - \Theta} \; \frac{c_n - \Theta}{\Theta - x}$$

From the above estimates the first term on the right hand side tends to zero since $\alpha_{\pm}(x) = 1$; in the second term the limit is $\phi(x)$, and the lemma is proved.

Remark (14). From Corollary 3 and Corollary 1 of section 1, $F_{(2)}(x)$ exists a.e. on $I(\lambda)$, and so a.e. on $E(A)$.

Theorem 2. If F is a $N^2_{s,g}$-primitive, F' exists a.e., as does $(F')'_{ap}$ which is, a.e., equal to $F_{(2)}$.

Proof. This follows from Remarks (1), (12), (13) and (14).

6. Some Generalizations. Much of the above discussion can be carried out with (A) replaced by

$$\forall x \in E,\ 0 \leq h \leq \varepsilon,\ |\Delta^2_s F(n,h)| \leq \psi(h) \qquad (\Psi)$$

where ψ is an even function, increasing with $|h|$. Thus lemma 1 remains valid if expressions such as $A(u-u_1)^2$, AD^2 (in (6), (6'), (7)) etc. are replaced by $\psi(u-u_1)$, $\psi(D)$ etc.

Similarly we can prove Lemma 2, but in (11) the expressions $A|K_{pj}|^2$ become $\psi(|K_{pj}|)$, and so the last term on the right hand side of (9) can only be written as $9 \sum_{p=0}^{q-1} 2^p \psi(\frac{b-a}{2^p})$.

However to obtain Theorem 1, ψ must be required to be continuous and zero at the origin, and satisfy the important condition (B). The following lemma gives four equivalent forms of condition (B).

Lemma 4. The following four conditions are equivalent:

(a) $\exists u_n,\ u_{n+1} \le u_n,\ n \ge 1,\ \lim_{n\to\infty} u_n = 0$ such that $\sum_{n=1}^{\infty} \frac{\phi(u_n)}{u_{n+1}} < \infty$;

(b) if $a > 0$ $\sum_{n=1}^{\infty} \phi(\frac{a}{n}) < \infty$;

(c) if $d > 1,\ a > 0$ $\sum_{n=1}^{\infty} d^n\, \phi(\frac{a}{n}) < \infty$;

(d) $\int_0^a \frac{\phi(u)}{u^2}\, du < \infty$.

Proof. See [11, II, pp.225-227].

Note that the convergence of the series in (a) does not imply the convergence of the series $\sum_{n=1}^{\infty} \frac{\phi(2u_n)}{u_{n+1}}$: it is this extra condition that is required to prove Lemma 3, when (15) becomes

$$F(x+h) - F(x) = hF'(x) + 4\delta\beta \int_0^{4h} \frac{\phi(t)}{t^2}\, dt.$$

It is trivial to remark that the function Au^2 satisfies condition (B).

Section 3. Applications of Denjoy's Index - II

1. Introduction. In this section the results from section 2 are used to calculate N_s^2-primitives. Finally some applications of Denjoy's index to first order symmetric derivative, and certain higher order derivatives, will be discussed.

2. Some Elementary Properties of the N_s^2-integral. Given $f \in N_s^2$, (or more

generally, $f \in N^2_{s,g}$), with N^2_s-primitive F, indefinite integral $\tilde{F}$, (section 2(1)) then the N^2_s-integral of f will be defined by Section 2(2) whatever the order of a, x and b, and will be zero if any two are equal. So if $f \in N^2_s(a,b)$ and c, d, e are any points of [a,b],

$$N^2_s - \int_{(c,d)}^{e} f = \int_{(c,d)}^{e} f = (e-c)(d-e)V_2(F;c,e,d), \quad c,d,e \text{ distinct}$$

$$= 0, \text{ otherwise} \qquad (1)$$

Remarks (1). $V_2(F;c,e,d)$, the second divided difference, is independent of the order of c, d and e.

(2). In Denjoy's notation $VR(F;c,e,d) = 2V_2(F;c,e,d)$ and $V(F;c,e,d) = (e-c)(d-e)(d-c)V_2(F;c,e,d)$, and so $(d-c) \int_{(c,d)}^{e} f = V(F;c,e,d) = F(c)(d-e) + F(e)(c-d) + F(d)(e-c)$.

Simple properties of V_2 give the following identities; let x,y,z, u and v be points of [a,b];

$$(z-x)(u-x) \int_{(x,z)}^{y} f + (y-x)(u-x) \int_{(x,u)}^{z} f + (y-x)(z-x) \int_{(x,y)}^{u} f = 0; \qquad (2)$$

$$(v-u)(z-x) \int_{(x,z)}^{y} f = (z-y)(v-x) \int_{(x,v)}^{u} f + (x-z)(v-y) \int_{(y,v)}^{u} f$$

$$+ (y-x)(v-z) \int_{(z,v)}^{u} f. \qquad (3)$$

Remark (3). If $\int_{(c,d)}^{e} f$ is known $\forall c \in C,\ d \in D,\ e \in E$ then using (2) and (3) we can evaluate $\int_{(x,y)}^{z} f \ \forall x,y,z \in C \cup D \cup E$.

If $f \in N^2_s(a,b)$ and $f \in N^2_s(b,c)$ it does not follow that $f \in N^2_s(a,c)$, as the following example shows. Define $f\colon [-\frac{2}{\pi}, \frac{2}{\pi}] \to \mathbb{R}$ by,

$$f(x) = 0, \ -\frac{2}{\pi} \le x \le 0$$

$$= x^{-3}\cos(x^{-2}), \ 0 < x \le \frac{2}{\pi};$$

$f \,\varepsilon\, N_s^2(-\frac{2}{\pi}, 0)$ with $\int_{(-\frac{2}{\pi},\, 0)}^{x} f = F_1(x) = 0, \ -\frac{2}{\pi} \le x \le 0$; and $f \,\varepsilon\, N_s^2(0,\frac{2}{\pi})$

with $\int_{(0,\, \frac{2}{\pi})}^{0} f = F_2(x) = x\cos(x^{-1}), \ 0 \le x \le \frac{2}{\pi},$

$$= 0, \ x = 0.$$

If then $f \,\varepsilon\, N_s^2(-\frac{2}{\pi}, \frac{2}{\pi})$, and $F(x) = \int_{(-\frac{2}{\pi},\, \frac{2}{\pi})}^{x} f, \ -\frac{2}{\pi} \le x \le \frac{2}{\pi}$, then F would be smooth and, for some A, B, C and D,

$$F(x) = F_1(x) + Ax + B, \ -\frac{2}{\pi} \le x \le 0,$$

$$= F_2(x) + Cx + D, \ 0 \le x \le \frac{2}{\pi};$$

simple calculations show that this is not possible; so $f \not\varepsilon\, N_s(-\frac{2}{\pi}, \frac{2}{\pi})$.

Lemma 1. If $f \,\varepsilon\, N_s^2(a,b)$ with indefinite integral F_1 and $f \,\varepsilon\, N_s^2(b,c)$ with indefinite integral F_2, then $f \,\varepsilon\, {}_s^2(a,c)$ iff $\lim\limits_{h\to 0+} \dfrac{F_1(b-h) + F_2(b+h)}{h}$ exists, λ say; and then the indefinite integral on [a,c] is F where,

$$F(x) = F_1(x) + \lambda\,\frac{(c-b)(x-a)}{c-a}, \ a \le x \le b,$$

$$= F_2(x) + \lambda\,\frac{(b-a)(c-x)}{c-a}, \ b \le x \le c.$$

Remarks. (4) The proof is by straightforward calculations; see Skvorcov [3].

(5) If $F_1'(b)$, $F_2'(b)$ exist then $\lambda = F_2'(b) - F_1'(b)$.

(6) If: $f:[a,b] \to \mathbb{R}$ is extended by periodicity to $[b,c]$, $c = 2b-a$, (and still called f), and if $f \in N_s^2(a,b)$, with indefinite integral F, then

$$f \in N_s^2(a,c) \text{ iff } \lim_{h\to 0+} \frac{F(a+h)+F(b-h)}{h} \text{ exists; see Skvorcov [2].}$$

(7) We say $f: [a,b] \to \mathbb{R}$ is M-integrable iff $f \in N_s^2(a,b)$, and the limit in Remark (6) exists, λ say; then $M - \int_a^b f = -\lambda$, ($= F'(b)-F'(a)$, if the derivatives exist); see Mařík.

(8) If $f \in N_s^2 \cap L(a,b)$ then $\int_{(a,b)}^x f$ is given by (4) of section 2.

3. Some Extensions of Some Elementary Formulas. Let $y_o,\ldots,y_p$ be any $p+1$ points at which F is defined; the definition of V_2 easily gives the following:

$$(y_p-y_1)(y_p-y_o)V_2(F;y_o,y_1,y_p) = F(y_p)-F(y_o)-(y_p-y_o)\frac{F(y_1)-F(y_o)}{y_1 - y_o}$$

$$= \sum_{i=1}^{p-1} (y_p-y_i)u_i; \tag{4}$$

$$-(y_{p-1}-y_o)(y_p-y_o)V_2(F;y_o,y_{p-1},y_p) = F(y_p)-F(y_o)-(y_p-y_o)\frac{F(y_p)-F(y_{p-1})}{y_p - y_{p-1}}$$

$$= \sum_{i=1}^{p-1} (y_o-y_i)u_i; \tag{5}$$

where $1 \leq i \leq p-1$,

$$u_i = (y_{i+1}-y_{i-1})V_2(F;y_{i-1},y_i,y_{i+1})$$

$$= \frac{F(y_{i+1})-F(y_i)}{y_{i+1}- y_i} - \frac{F(y_i)-F(y_{i-1})}{y_i - y_{i-1}}. \tag{6}$$

Now consider a subdivision of $[u,v]$, $u = y_o < \dots < y_m = y < \dots < y_m = v$, apply (5) to $y_o,\dots,y_m$, and (4) to $y_m,\dots,y_n$ and add to get:

$$\frac{F(v)-F(y)}{v - y} - \frac{F(y)-F(u)}{y - u} = \frac{1}{y-u}\sum_{i=1}^{m-1} (u-y_i)u_i + \frac{1}{v-y}\sum_{i=m+1}^{n-1} (v,y_i)u_i + u_m. \tag{7}$$

Remark (9). If z_1, z_2 are two further points where F' exists then from (6) we easily get:

$$F'(z_2) - F'(z_1) = w_1(z_1) + \sum_{=1}^{p-1} u_1 + w_2(z_2), \tag{8}$$

where

$$w_1(z_1) = \frac{F(y_o)-F(z_1)}{y_o - z_1} - F'(z_1),$$

$$w_2(z_2) = F'(z_2) - \frac{F(z_2)-F(y_p)}{z_2 - y_p}.$$

For later applications it is important to extend (4), (5) and (7) to obtain expressions where the terms on the various left hand sides are evaluated not at points of $y_o,\dots,y_p$ but at limit points of such a sequence. Let $u < y_i < v$, $y_{i-1} < y_i$, $i \in Z$, $\lim_{i\to-\infty} y_i = u$, $\lim_{i\to\infty} y_i = v$; (the case where only one of u, v is a limit point can be considered similarly.) Choose three other points y', y, y'' with $y_r < y' < y_{r+1} < y_s < y < y_{s+1} < y_t < y'' < y_{t+1}$; (as we want to let $y' \to u$, $y'' \to v$, missing the terms of the sequence avoids minor difficulties.)

Define $C_-(y',y)$ by the right hand side of (5) using y', $y_{r+1},\dots,y_s,y$, and $C_+(y,y'')$ by the right hand side of (4) using y, $y_{s+1},\dots,y_t,y''$:

$$C_-(y',y) = (y'-y_{r+1})u_+(y') + \sum_{i=r+2}^{s-1} (y'-y_i)u_i + (y'-y_s)u_-(y); \tag{9}$$

$$C_+(y,y'') = (y''-y_{s+1})u_+(y) + \sum_{i=s+2}^{t-1} (y''-y_i)u_i + (y''-y_t)u_-(y''); \tag{10}$$

where $u_+(y')$ is u_{r+1} with y_r replaced by y', $u_-(y)$ is u_s with y_{s+1} replaced by y, $u_+(y)$ is u_{s+1} with y_s replaced by y, $u_-(y'')$ is u_t with y_{t+1} replaced by y''. (The suffix in C_- indicates the left point y' occurs in the expression; in u_- it indicates that in (6) the first term is altered, etc.)

From (5), and (4),

$$C_-(y',y) = F(y) - (y-y') \frac{F(y)-F(y_s)}{y - y_s}, \tag{11}$$

$$C_+(y,y'') = F(y'') - F(y) - (y''-y) \frac{F(y_{s+1})-F(y)}{y_{s+1} - y}. \tag{12}$$

Assume now that F is left continuous at v, and right continuous at u then (11) and (12) show that the following limits exist; a fact that is not obvious from the definitions of $C_\pm$, (9) and (10);

$$\begin{aligned} \mu_+(u,y) &= \lim_{y'\to u+} C_-(y',y), \\ \mu_-(y,v) &= \lim_{y''\to v-} C_+(y,y''). \end{aligned} \tag{13}$$

From (11), (12) and (13)

$$F(y) - F(u) - (y-u) \frac{F(y)-F(u)}{y - u} = \mu_+(u,y) \tag{14}$$

$$F(v) - F(y) - (v-y) \frac{F(v)-F(y)}{v - y} = \mu_-(y-v) \tag{15}$$

Remark (10). (14) and (15) can be regarded as extensions of (4) and (5). The left hand sides, given the definitions of $\mu_\pm$, (13), can be found once all the u_i are known, $i \in Z$, as well as $u_\pm(y)$, $u < y < v$, provided F is right continuous at u, and left continuous at v.

Now define

$$c(u,y,v) = \frac{\mu_-(y,v)}{v - y} - \frac{\mu_+(u,y)}{y - u} + u(y),$$

where

$$u(y) = \frac{F(y_{s+1})-F(y)}{y_{s+1}-y} - \frac{F(y)-F(y_s)}{y-y_s},$$

then from (14) and (15)

$$\frac{F(v)-F(y)}{v-y} - \frac{F(y)-F(u)}{y-u} = c(u,y,v). \tag{17}$$

Remark (11). This is the required generalization of (7). The left hand side of (17) is $(v-u)V_2(F;u,y,v)$, and (17) shows how this quantity, (or equivalently $\int_{(u,v)}^{y} f$,) can be calculated from the knowledge of similar quantities in $]u,v[$, provided the limits (13) can be calculated; and for this the continuity of F at u and v will suffice.

If we now assume the existence of $F'_+(u)$ and $F'_-(v)$ then, from (17) the following limits exist:

$$\begin{aligned} \mu(u,v) &= \lim_{y\to u+} c(u,u,v), \\ \nu(u,v) &= \lim_{y\to v-} c(u,y,v), \end{aligned} \tag{18}$$

and have the values,

$$\mu(u,v) = \frac{F(v)-F(u)}{v-u} - F'(u),$$

$$\nu(u,v) = F'(v) - \frac{F(v)-F(u)}{v-u}.$$

Hence the quantities

$$\begin{aligned} s(u,v) &= \frac{1}{2}(\mu(u,v) - \nu(u,v)), \\ &= \frac{F(v)-F(u)}{v-u} - \frac{1}{2}\{F'(u)+F'(v)\}, \end{aligned} \tag{19}$$

$$\begin{aligned} t(u,v) &= \mu(u,v) + \nu(u,v), \\ &= F'(v) - F'(u), \end{aligned}$$

can be calculated, as suggested in Remarks (10) and (11), from knowledge of various quantities in $]u,v[$, subject to the extra condition that $F'_+(u)$, $F'_-(v)$ exist. The quantities (19) are of importance for our calculation; see section 8 below.

Remark (12). It should be emphasized that in all cases the limits need to be evaluated; they cannot be found by suitable substitutions into the definitions. Thus $\mu_+(u,y)$ is given by (16) from (11) and not from (11) as $C_-(u,y)$: see Denjoy [11, pp.293-305].

4. The First Set of Calculations. The process whereby $\int_{(c,d)}^{x} f$ can be calculated, $a \leq c \leq x \leq d \leq b$, for a given $f \in N^2_{s,g}$ is called the second order symmetric totalisation of f. If, given $f: [a,b] \to R$ the process works we will say that $f \in T^2_s(a,b)$, and write the total as $T^2_s\text{-}\int_{(c,d)}^{x} f$, or just $\int_{(c,d)}^{x} f$ when there is no ambiguity. We wish to give the conditions under which this totalisation is possible, and then check that if $f \in N^2_{s,g}(a,b)$ the conditions hold and $N^2_s\text{-}\int_{(c,d)}^{x} f = T^2_s\text{-}\int_{(c,d)}^{x} f$. If, at any stage of the process, this is the case then the process will be said to be satisfactory, at that stage.

Remark (13). This approach to T^2_s-totalisation is modelled on that for the first order totalisations, see for instance Bullen [3]. However we will have to change our approach and redefine what we mean by $f \in T^2_s(a,b)$; see paragraph 5.

Condition 0: (C.0). Let $S = \{x; a \leq x \leq b$, f is not L-integrable at $x\}$, then S, which is closed, is nowhere dense in [a,b].

Operation 1: (0.1). If $a \leq c \leq x \leq d \leq b$ and $f \in L(c,d)$ then $T^2_s\text{-}\int_{(c,d)}^{x} f$ is given by the right hand side of (4), section 2, (with, of course, a,b replaced by c,d).

This operation calculates the total, by repeated Lebesgue integration, on any [c,d] with $[c,d] \cap S = \phi$.

Remarks (14). (C.0) is made unnecessary by (C.3) below: see Remark (18).

(15). If $f \in N^2_{s,g}$, $P \neq \phi$, perfect, then the set of points $x \in P$, in the neighbourhood of which $A(x) = \max_h \frac{|\Delta^2_s F(x,h)|}{h^2}$ is not bounded, S(P) say, is a closed nowhere dense subset of P. If then $[c,d] \cap S(P) = \phi$, A(x) is bounded on $P \cap [c,d]$. This shows that (C.0) is satisfied, and that at this stage (0.1) makes the process satisfactory.

Condition 1: (C.1). If $T^2_s\text{-}\int_{(c,d)}^{x} f$ is known for all c,d,x, $u < c \leq x \leq d < v$, then $\forall y$, $u \leq y \leq v$ the limit as $x \to y$, $c \to u+$, $d \to v-$ of $\int_{(c,d)}^{x} f$ exists.

Operation 2: (0.2). Evaluate the limit in (C.1) as $T^2_s\text{-}\int_{(u,v)}^{y} f$.

This operation will give the total on all closed contiguous intervals of the set S of (C.0); further since $N^2_{s,g}$-primitives are continuous the process is satisfactory.

Condition 2: (C.2). If $F_1(x) = T_s^2\text{-}\int_{(c,d)}^{x} f$, $c \leq x \leq d$, $F_2(x) = T_s^2\text{-}\int_{(d,e)}^{x} f$,

$$d \leq x \leq e, \text{ then } \lim_{h\to 0+} \frac{F_1(d-h)+F_2(d+h)}{h} \text{ exists.}$$

Operation 3: (0.3). Evaluate the limit in (C.2) as λ, say, and define

$$T_s^2\text{-}\int_{(c,e)}^{d} f = \lambda \frac{(e-d)(d-c)}{(e-c)} .$$

Once (0.3) has been performed then $\int_{(c,e)}^{x} f$, $c \leq x \leq e$ can be found from F_1, F_2 and $\int_{(c,e)}^{d} f$ by a use of (2). Further since $N_{s,g}^2$-primitives are smooth Lemma 1 shows the process is satisfactory.

Using (0.1)-(0.3) the total can be found on all the closed contiguous intervals of the perfect kernel of the set S of (C.0), by a process common to all totalisations; see Bullen [3]. This ends the first set of calculations that can be used to solve the following problem:

Problem 1: (P.1). If the total has been calculated on the closed subintervals of the contiguous intervals of a closed set, find the total on the closed contiguous intervals of the perfect kernel of that set.

Remark (16). If the process is satisfactory up to the given calculations of (P.1), then it is so up to the final calculation in (P.1).

5. The Second Set of Calculations. Preliminaries. Totalisation consists of finding a decreasing sequence of perfect sets P_α, $0 \leq \alpha < \Omega$, each nowhere

dense in its predecessors, and for each of which the total is known on its closed contiguous intervals; then after a countable number of steps the total on [a,b] will be known; see Bullen [3]. The discussion in paragraph 4 gives the first two perfect sets of the sequence; P_o = [a,b], P_1 = the perfect kernel of the set S of (C.0), and so it is necessary to solve the following problem:

Problem 2: (P.2). If the total is known on the closed contiguous intervals of a perfect set, find the total on the closed sub-intervals of the contiguous intervals of some nowhere dense closed subset.

If we have solved (P.2) then by (P.1) we can proceed to the perfect kernel of the closed subset in (P.2); if then the perfect set in (P.2) is P_1, we have arrived at P_2, and the process can be completed.

Remark (17). Of course if the process is satisfactory up to the given calculations of (P.2) it must remain so after (P.2) has been solved.

Suppose that (P.2) has been solved for perfect sets of definite type, (see section 1) and let $P \varepsilon D_4$. If H is the set of Section 1, Theorem 4, [c,d] one of its contiguous intervals, then on this interval P consists of a countable number of isolated portions of definite type that can only accumulate, if at all, at c and, or, d. Then (P.2) is solved for P by the solution, assumed, for perfect sets of definite type; the closed nowhere dense subset of (P.2) for P is H, together with the closed nowhere dense subsets of (P.2) on each of the portions of definite type, on each [c,d]; see Denjoy [11, p.324].

It remains then to consider (P.2) for perfect sets of definite type.

For this it is necessary to change our definition of the class of T_s^2-totalisable functions from a constructive one to a descriptive one. We will introduce a class of functions on $[a,b]$, $\Gamma_s^2(a,b)$, and $F \in \Gamma_s^2$ iff F satisfies (C.1)-(C.4) below; it will be seen that if $F \in \Gamma_s^2$ then $(F')'_{ap}$ will exist a.e. and then if $(F')'_{ap} = 0$ a.e. F is constant; (see Remark (28) below).

Definition 1. If $f : [a,b] \to R$ then $f \in T_s^2(a,b)$ iff $\exists$ $F \in \Gamma_s^2(a,b)$ and

$(F')'_{ap} = f$ a.e.; and then $T_s^2\text{-}\int_{(c,d)}^{e} f$, $a \leq c, d, e \leq b$, is given by the right hand side of (1).

If now $f \in T_s^2$ the T_s^2-totalisation process will be possible, using (0.1)-(0.3) and (0.4)-(0.9) to be defined, and will give $T_s^2\text{-}\int_{(c,d)}^{e} f$ as defined in Definition 1. It will be clear from the discussion, and from section 2 that if $f \in N_{s,g}^2$ then $f \in T_s^2$ and the integrals will be the same.

To ensure that the preceeding discussion will apply in this new situation we state the first two conditions for Γ_s^2, conditions corresponding to (C.1) and (C.2) above.

Condition 1: (C.1). If $F \in \Gamma_s^2(a,b)$ then F is continuous on $[a,b]$.

Condition 2: (C.2). If $F \in \Gamma_s^2(a,b)$ then F is smooth on $]a,b[$.

6. Problem Two for Perfect Sets of Type One. If $P \in D_1$ then for some λ_o $P = I(P,\lambda)$, $\lambda > \lambda_o$; so the next character of Γ_s^2 will concern behaviour

relative to $I(P,\lambda)$ sets for arbitrary perfect subsets of $[a,b]$. The form of (C.3) below is suggested by properties discussed in section 2.5.

Condition 3: (C.3). If $F \in \Gamma_s^2$, $P \neq \phi$, perfect then $\exists$ closed nowhere dense subset $S = S(P)$ such that $\forall Q \neq \phi$, a closed portion of P, $Q \cap S = \phi$, $\lambda > 1$, F' exists and is AC on $I(Q,\lambda)$.

As in Section 2 Theorem 2 (C.3) implies that $(F')'_{ap}$ exists a.e. in $[a,b]$ and that $F \in ACG$. Further while the quantities s_n, t_n of section 2 (9) do not necessarily satisfy section 2 (20), nevertheless both $\sum|s_n|$ and $\sum|t_n|$ converge.

Remarks (18). By applying (C.3) with $P = [a,b]$ we see that (C.3) implies (C.0).

(19). If $f \in N_{s.g}^2$ then the set $S(P)$ of (C.3) is taken to be the set of Remark (15), and the discussion in section 2 shows (C.3) holds.

We will now introduce two operations which will solve the following problem:

Problem 3: (P.3). Let $I(Q,\lambda)$ be as in (C.3) and suppose the total is known on all of its closed contiguous intervals. To find the total on its containing interval: ($f \in T_s^2$, of course).

Let $[c,d]$ be the containing interval, $[c_n,d_n]$ a closed contiguous interval in $[c,d]$, $F_n(x) = \int_{(c_n,d_n)}^{x} f$, $c_n \leq x \leq d_n$.

Remark (20). It is sufficient for (P.3) to obtain $\int_{(u,v)}^{x} f, u,v,x \in I(Q,\lambda)$, since an application of (2) will then evaluate the total for any three points in $[c,d]$.

Operation 4: (0.4). With the above notation evaluate the limits

$$\mu_n = \lim_{x\to c_n} \frac{F_n(x)}{c_n - x}, \quad \nu_n = \lim_{x\to d_n} \frac{F_n(x)}{x-d_n},$$

and calculate $s_n = \frac{1}{2}(\mu_n - \nu_n)$, $t_n = \mu_n + \nu_n$, $n \geq 1$.

Since $f \in T_s^2$ each F_n differs from $F \in \Gamma_s^2$ by an affine function and so, by (C.3), μ_n and ν_n exist being just:

$$\mu_n = \frac{F(d_n)-F(c_n)}{d_n - c_n} - F'(c_n),$$

$$\nu_n = F'(d_n) - \frac{F(d_n)-F(c_n)}{d_n - c_n},$$

and then,

$$s_n = \frac{F(d_n)-F(c_n)}{d_n - c_n} - \frac{1}{2}\{F'(c_n) + F'(d_n)\},$$

$$t_n = F'(d_n)-F'(c_n);$$

see section 2 (9), and (19).

The final operation to solve (P.3) is justified by the following calculations, based on the properties of F given by (C.3), see Denjoy [11; pp.268-269].

If $u,v \in I(Q,\lambda)$ then,

$$F(v) - F(u) = \int_u^v F' 1_{I(Q,\lambda)} + (u \textstyle\sum v)\{F(d_n)-F(c_n)\};$$

and if Φ, ϕ are as in section 2 after Corollary 3 and (18).

$$F'(v) - F'(u) = \int_u^v \phi \, 1_{I(Q,\lambda)} + (u \textstyle\sum v)t_n$$

$$= \int_u^v X ;$$

$$F(v)-F(u)-(v-u)F'(u) = \int_u^v \{F'(t)-F'(u)\} 1_{I(Q,\lambda)}(t)dt$$

$$+ (u \sum v) \ [(d_n-c_n)\{\tfrac{1}{2}(F'(c_n)+F'(d_n))-F'(u)+s_n\}]$$

$$= \int_u^v \int_u^s X(t)dtds;$$

where

$$X(t) = \phi(t), \ t \in I(Q,\lambda)$$

$$= \frac{t_n}{d_n - c_n} + \frac{s_n}{(d_n - c_n)^2}\{\tfrac{1}{2}(c_n + d_n)-t\}, \ c_n < t < d_n, \ n \geq 1. \qquad (20)$$

So, as in (4) of section 2, if $u,v,x \in I(Q,\lambda)$,

$$(x-u)(v-x)V_2(F;u,x,v) = \frac{1}{(u-v)}\{\int_u^x (t-u)X(t)dt + \int_x^v (v-t)X(t)dt\}.$$

(21)

Since, see section 2, a.e. on $I(Q,\lambda), \phi = (F')'_{ap}$ we can, and will, replace in the definition of X, ϕ by f and then the following operation, given Remark (20), completes the solution of (P.3).

Operation 5: (0.5). With the above notation if $u,v,x \in I(Q,\lambda)$ define

$$T^2_s\text{-}\int_{(u,v)}^x f \text{ by the right hand side of (21).}$$

Remarks (21). It is not difficult to see, from (20) and (21), that (0.5) consists of Lebesgue integration of $tf(t)$ and $f(t)$ on a nowhere dense perfect set, together with the summing of an absolutely convergent series; Denjoy [11,p.283].

(22) The constructive approach of paragraph 4 could have been continued to this point by replacing (C.3) by the following: (a) $f \in L(I(Q,\lambda))$, (b) the limit in (0.4) exists, (c) $\sum|s_n|$ and $\sum|t_n|$ converge.

However the discussion is better motivated from the constructive point of view, and the change introduced is essential for the next section.

Clearly if $P \in D_1$ then the solution of (P.3) is the solution of (P.2) since $I(P,\lambda) = P$, $\lambda > \lambda_o$.

Remark (23). When $f \in N_s^2$ the set $S(P)$, see Remark (19), is not determined by our knowledge of P and $f = D_s^2 F$ on P; see Denjoy [11,pp.319-323]. So the solution of (P.2) must proceed a little differently. Decompose P into an infinity of closed isolated portions Q_n, $\lim_{n\to\infty}|Q_n| = 0$, each point of P being in an infinity of the Q_n; attempt the above calculations with $Q_1, Q_2, \ldots$. While these calculations may fail for some Q_n, (if $Q_n \cap S(P) \neq \phi$), they must succeed on many as $S(P)$, although unknown, is nowhere dense. In this way we find simutaneously both $S(P)$ and the totals on the closed sub-intervals of its contiguous intervals. This remark applies to other stages but will not be repeated.

7. Problem Two for Sets of Type 2. If $P \in D_2$ then $\forall \lambda$, $P = J(P,\lambda)$; and so the next condition is related to $J(P,\lambda)$ for any perfect subset of [a,b].

Condition 4: (C.4). If $F \in \Gamma_s^2$, $P \neq \phi$, perfect $\exists$ closed nowhere dense subset $C = C(P)$ such that $\forall Q \neq \phi$, a closed portion of P, $Q \cap C = \phi$, if $K = [u,v]$ is a λ-maximal segment of Q, $\lambda > 1$ and if

$$\omega_\lambda(K) = \frac{\Delta_s^2 F(u,|K|)+\Delta_s^2 F(v,|K|)}{|K|},$$

$$\sigma_\lambda = \max_K \left| \frac{\Delta_s^2 F(u,|K|)}{|K|} \right|,$$

then

(a) $\sum_K \omega_\lambda(K) < \infty$,

(b) $\lim_{\lambda\to\infty} \max_{x,y} |x \sum_K y\, \omega_\lambda(K)| = 0$,

(c) $\exists$ M such that $\forall \lambda$, $\sigma_\lambda < M$.

Remark (24). This completes the definition of Γ_s^2 and it is easily see that if F, G ε Γ_s^2 then aF + bG ε Γ_s^2.

(25). If f ε $N^2_{s,g}$ we take C(P) to be the set S(P) of Remark (15); then since $\frac{\Delta_s^2 F(x,h)}{h^2}$ is bounded on Q (C.4) is easily seen to be satisfied.

To solve (P.2) for P ε D_2 it suffices to calculate $\int_{(x_o,x_1)}^{x} f$ for x_o, x_1 is one contiguous interval of Q, x is another since then by use of (2), (3) and (0.2), Q being nowhere dense, the total can be found on the containing interval of Q. So for simplicity let us consider a P ε D_2, the total being known on its closed contiguous intervals, P $\subset$ [c,d], with [c',d'] its containing interval, c < c' < d' < d, $K_n = [u_n, v_n]$ its disjoint λ-maximal segments, $1 \leq n \leq m$. We will assume that $c' = u_1 < v_1 < \ldots < v_m = d'$ and that λ is large enough that each K_n is so isolated from its neighbour that $v_n + |K_n| < u_{n+1} - |K_{n+1}|$, $1 \leq n \leq m-1$. If $c \leq x_o < x_1 < c'$, $d' < x \leq d$ we can also assume, by taking λ large enough that $x_1 < u_1 - |K_1|$, $v_m + |K_m| < x$.

Now define x_k, $0 \leq k \leq 4m+2$ as follows: x_o, x_1 as above, $x_{4m+2} = x$, and if $1 \leq n \leq m$, $x_{4n-2} = u_n - |K_n|$, $x_{4n-1} = u_n$, $x_{4n} = v_n$, $x_{4n+1} = v_n + |K_n|$. Then, from above, if λ is large enough, $c \leq x_o < x_1 < \dots < x_{4m+2} \leq d$. If then $f \in T_s^2$ and $F \in \Gamma_s^2$ is given by Definition 1, we have from (4) and (6) that

$$(x-x_o)(x-x_1)V_2(F;x_o,x_1,x) = \sum_{k=1}^{4m+1} (x-x_k)u_k, \tag{22}$$

$$u_k = (x_{k+1}-x_{k-1})V_2(F;x_{k-1},x_k,x_{k-1}),\ k \equiv 2,1 \pmod 4 \tag{23}$$

$$= \frac{\Delta_s^2 F(u_k,|K_n|)}{|K_n|},\ k = 4n-1,$$

$$= \frac{\Delta_s^2 F(v_n,|K_n|)}{|K_n|},\ k = 4n \tag{24}$$

(P.2) will be solved if we can calculate the left hand side of (22) from the known totals on the closed contiguous intervals of P, $[x_k,x_{k+3}]$, $k \equiv 0 \pmod 4$, $1 \leq k \leq 4m-1$, $[c,x_3]$ and $[x_{4m},d]$. Since each of these totals differ from F by an affine function they have the same second divided differences as F, and so the terms on the right hand side arising from u_k given by (23) can be determined; call their sum $\sum_1$. The remaining terms, those with u_k given by (24), cannot be calculated as the points at which the second divided differences are calculated lie in two different closed contiguous intervals of P; call this sum $\sum_2$.

$$\sum\nolimits_2 = \sum_{\substack{k=1 \\ k\equiv 0,-1 \pmod 4}}^{4m+1} (x-x_k)u_k$$

$$= \sum_{\substack{k=1 \\ k\equiv -1 (\mathrm{mod}\ 4)}}^{4m+1} \{(x-x_k)u_k + (x-x_{k+1})u_{k+1}\}$$

$$= \sum_{n=1}^{m} \{(x-x_n)\omega_\lambda([u_n,v_n]) - |K_n|u_{4n}\}$$

$$\leq (x-c')\max_{u,v} |(u \textstyle\sum v)\omega_\lambda([u_n,v_n]) + \frac{d'-c'}{\lambda}\sigma_\lambda, \tag{25}$$

so by (C.4) $\lim_{\lambda\to\infty} \sum_2 = 0$. Equivalently, by (C.4)

$$(x-x_o(x-x_1)V_2(F;x_o,x_1,x_2) = \lim_{\lambda\to\infty} \sum_1$$

Operation 6: (0.6). With the above notation evaluate $T_s^2 - \int_{(x_o,x_1)}^{x} f$ as

$$\lim_{\lambda\to\infty} \sum_1 = \lim_{\lambda\to 0} \sum_{\substack{k=1 \\ k\equiv 1,(\mathrm{mod}\ 4)}}^{4m+1} \{(x-x_k)u_k + (x-x_{k+1})u_{k+1}\}.$$

Remark (26). If $F \in \Gamma_s^2$ then $F \in C \cap AGC$ and F' exists a.e. Hence F can be found from F' by a simple totalisation. In the present case of (P.2), $P \in D_2$, $|P| = 0$ and so knowing the total on all the closed contiguous intervals of P, F' is known a.e. However our information does not give $F'(y) - F'(x)$, if x,y lie in different closed contiguous intervals of P; for the known totals each differ from F by an affine function. An argument similar to the above, but using (8) rather than (4), can be used to obtain $F'(y) - F'(x)$, if we introduce a suitable variant of (0.6). Then the totalisation can be completed in the way suggested; see Denjoy [11; pp.286-287].

Remark (27). It is the need to know that $\lim_{\lambda\to\infty} \sum_2 = 0$, and not just that it exists, that caused us to introduce the class Γ_s^2.

8. Problem Two for Perfect Sets of Type Three. If $P \in D_3$ we will assume for simplicity that (C.3) and (C.4) hold on P; that is, P is identified with the closed portion on which both (C.3) and (C.4) hold. Since $\Omega(P)$ is dense in P, see section 1, it is sufficient by (0.2) to calculate $\int_{(u,v)}^{x} f$ for u,v,x in $\Omega(P)$. However if $u,v,x \in \Omega(P)$ then for all large enough λ, $u,v,x \in I(P,\lambda)$, and so it suffices to approximate the total for $u,v,x \in I(P,\lambda)$, provided the error in this approximation tends to zero as $\lambda \to \infty$.

Let $P = I(P,\lambda) \cup J(P,\lambda)$, [c,d] a closed contiguous interval of $I(P,\lambda)$. The discussion of paragraph 6 shows that an approximation to $\int_{(u,v)}^{x} f$, $u,v,x \in I(P,\lambda)$ can be found if suitable approximations to the s, and t of (0.4) can be obtained. Of course if $[c,d] \cap J(P,\lambda) = \phi$ then s and t can be found exactly, by (0.4).

Suppose $[c,d] \cap J(P,\lambda)$ consists of a finite number of λ-maximal portions, (in a containing interval [c',d'], $c < c' < d' < d$), then we attempt to calculate F'(d)-F'(c), and so t, as in paragraph 7, using (8) as suggested in Remark (26). However we cannot, as in (0.6), let $\lambda \to \infty$, since obviously c and d depend on λ. Instead we write $t = t_1 + t_2$, where t_1 is known, arising from terms in (8) calculated from the same contiguous interval, t_2 unknown, but by (C.4) we can show that t_2 satisfies an inequality similar to (25). In a similar way we can write $s = s_1 + s_2$, s_1 known, s_2 unknown but bounded, as t_2, by an inequality similar to (25); see Denjoy [11;pp.293-295].

It remains to consider the case $[c,d] \cap J(P,\lambda)$ containing an infinity of λ-maximal segments $K_n = [u_n, v_n]$, $n \in Z$, with $v_n < u_{n+1}$, $n \in Z$, $\lim_{n\to-\infty} v_n = c$, $\lim_{n\to\infty} u_n = d$; the case where only one of c, or d, is a point of accumulation can be treated similarly. As in paragraph 7 we define a sequence x_k by $x_{4n-2} = u_n - |K_n|$, $x_{4n-1} = u_n$, $x_{4n} = v_n$, $x_{4n+1} = v_n + |K_n|$, $n \in Z$; if $\lambda > 2$ this is an increasing sequence, and $\lim_{k\to-\infty} x_k = c$, $\lim_{k\to\infty} x_k = d$. Now let $x' < x < x''$ be three points not lying in any K_n, and distinct from all the x_k; precisely we assume $x_{4p+1} < x' < x_{4p+2}$, $x_{4q+1} < x'' < x_{4r+2}$.

With obvious changes in notation we can apply (9) and (10) to this sequence, and, as above the quantities $C(x',x)$, $C_+(x,x'')$ can be written as $C_\pm = C_{\pm,1} + C_{\pm,2}$, where the $C_{\pm,1}$ can be calculated from known information, and $C_{\pm,2}$ can be estimated by inequalities similar to (25).

Since $f \in T_s^2$ we have that $F \in \Gamma_s^2$ and of course c, d $\in I(P,\lambda)$ so that conditions of paragraph 3 apply, see Remarks (10), (11); hence the various limits of that paragraph exist; $\mu_+(c,x)$, $\mu_-(x,d)$, $c(x,x,d)$, $\mu(c,d)$, $\nu(c,d)$, $s(c,d)$, $t(c,d)$, see (13), (16), (18) and (19). Estimates of a simple nature show that the similar limits $\mu_{\pm,2}$, c_2, μ_2, ν_2, s_2, t_2 also exist; see Denjoy [11,p.307-311]. This justifies the following two operations.

Operation 7: (0.7). With the above notations evaluate

$$\mu_{+,1}(c,x) = \lim_{x'\to c+} C_{-,1}(x'x),$$

$$\mu_{-,1}(x,d) = \lim_{x''\to d-} C_{+,1}(x,x''),$$

and calculate $c_1(c,x,d)$.

Operation 8: (0.8). With the above notations evaluate

$$\mu_1(c,d) = \lim_{x\to c} c_1(c,d,x),$$

$$\nu_1(c,d) = \lim_{x\to d} c_1(e,x,d),$$

and calculate $s_1(c,d)$, $t_1(c,d)$.

In addition calculations similar to those giving (25) give

$$|s_2(c,d)|,\ |t_2(c,d)| \le M\,\frac{(d-c)}{\lambda}, \tag{26}$$

for some constant M; see Denjoy [11,p.314].

Carrying out these calculations for each such closed contiguous interval of $I(P,\lambda)$ enables us to define X_1 as in (20) but with s_n, t_n replaced by $s_{n,1}$, $t_{n,1}$, (and ϕ by f of course). Calculate the right hand side of (21) using X_1 and call this $T^2_{s,1}\text{-}\int_{(u,v)}^{x} f$, where u, v, $x \in I(P,\lambda)$.

Define X_2 by (20) with s_n, t_n replaced by $s_{n,2}$, $t_{n,2}$ and ϕ replaced by 0; use the right hand side of (21) with X_2 to calculate a quantity that by estimate (26) can be shown to be bounded by $\frac{K(b-a)^3}{\lambda}$; Denjoy [11,p.316].

Operation 9: (0.9). If u,v,x are any three points of $\Omega(P)$

$$\text{Calculate } T^2_s\text{-}\int_{(u,v)}^{x} f \text{ as } \lim_{\lambda\to\infty} T^2_{s,1}\text{-}\int_{(u,v)}^{x} f.$$

This by previous remarks complete the discussion of (P.2) for sets of type 3 and so of T^2_s-totalisation.

Remarks (28). Using the various techniques of T^2_s-totalisation it is possible to prove that if $F \in \Gamma^2_s$, $(F')'_{ap} = 0$ a.e. then F is a constant; see Denjoy [11;pp.478-480].

(29). A special subclass of N^2_s consists of the sums of everywhere convergent trigonometric series. Even in this special case T^2_s-totalisation, which calculates the coefficients of the series from the sum, cannot be shortened. Precisely, $\forall\alpha$, $\alpha < \Omega$, $\exists$ such an f for which the process has order type α, and $\forall\beta$, $\beta < \alpha$, all nine operations are used beyond the β stage; see Denjoy [11,pp.483-495].

9. The First Order Symmetric Derivative. If F: $[a,b] \to R$ is continuous and $D_s F = f$ at all points of $]a,b[$, where

$$D_s F(x) = \lim_{h\to 0} \frac{F(x+h)-F(x-h)}{2h},$$

then we say $F \in N_s$, $N_s\text{-}\int_a^b f = F(b)-F(a)$. In this section we will discuss a totalisation process that will calculate $F(x)-F(a)$, from a given $f \in N_s$, or even more generally $f \in N_{s,g}$ meaning $-\infty < \underline{D}_s F \leq f \leq \overline{D}_s F < \infty$, F continuous. As in section 2, if F $N_{s,g}(a,b)$ then $[a,b]$ is the union of a countable collection of closed sets of the type $E = E(A)$, where

$$\forall x \in E,\ 0 \leq h \leq \varepsilon,\ |F(x+h) - F(x-h)| \leq 2Ah. \qquad \text{(A)}$$

Proceeding as in, Section 2 Lemma 1, (see Denjoy 12-II, and 11, pp.235-237), if Θ_1, $\Theta_2 = \Theta_1 + k$ are two points of E(A), $u = u_o = \Theta + h$, any other point, and if, for $m \in Z$, u_n is defined by (SS), see section 2, and if $|h| > 2|k|$, $|h+2mk| < 2|k|$, $h(h+2mk) > 0$, then

$$F(u_{2m}) = F(u) + 2\delta A \sum_{r=0}^{2|m|-1} (|h| - r|k|)$$

$$= F(u) + 4\delta A \frac{h^2}{k}, \ (\delta^2 < 1).$$

If now x, $x+k_n$ E(A), $2|k_{n+1}| < |k_n| < \beta|k_{n+1}|$, $\lim_{n\to\infty} k_n = 0$, $|h| < 4|k_1|$, then as in lemma 3 of section 2

$$F(x+h) = F(x) + \delta CA\beta h,$$

where $|\delta| < 1$, and C does not depend on x,h,β or A.

Lemma 2. $\forall x \in E(A)$ of finite index, α say, $F(x+h) = F(x) + 2\delta C\alpha h$ and

$$|\overline{\underline{D}}_{\pm} F(x)| < 2C\alpha.$$

Corollary. If $f \in N_{s,g}$ with N_s-primitive F then F' exists a.e.

Conditions 1, 2 and 3 below will define a linear class of functions Γ_s and if $F \in \Gamma_s$, F' exists a.e., and further if F' = 0 a.e. F is a constant.

Definition 2. If f: $[a,b] \to R$ then $f \in T_s(a,b)$ iff $\exists\, F \in \Gamma_s$ and F' = f a.e.;

$$\text{then } T_s - \int_a^x f = F(x), - F(a),\ a \leq x \leq b.$$

Condition 1: (C.1). If $F \in \Gamma_s(a,b)$ then F is continuous in [a,b].

Condition 2: (C.2). If $F \in \Gamma_s$, $P \neq \phi$, perfect, $\exists$ closed nowhere dense subset $S = S(P)$ such that $\forall Q \neq \phi$, a closed portion of P, $Q \cap S = \phi$, $\lambda > 1$, F is AC* on $I(Q,\lambda)$.

Condition 3: If $F \in \Gamma_s$, $P \neq \phi$, perfect, closed nowhere dense subset $C = C(P)$, such that $Q \neq \phi$, a closed portion of P, Q C = ϕ, $\lambda > 1$, if L is a λ-maximal portion of Q, K = [u,v] the associated λ-maximal segment, define for $x \in L$, $|K| \leq h \leq 2|K|$,

$\omega_\lambda(K) = \max\limits_{x,h} |F(x+h)-F(x-h)|$, then (a) $\sum\limits_K \omega_\lambda(K) < \infty$,

(b) $\lim\limits_{\lambda\to\infty} \{\max\limits_{x,y} (x\int y)\omega_\lambda(K)\} = 0.$

If $f \in N_{s,g}$ then $f \in T_s$; (C.1) is trivial and for (C.2), and (C.3), we take the sets S(P), and C(P), to be the closed nowhere dense subset of P where $\dfrac{F(x+h)-F(x-h)}{2h}$ is not bounded. Then in (C.2) we note that on $I(Q,\lambda)$ the index is bounded and so by Lemma 2, $f \in L(I(Q,\lambda))$; also by Lemma 2, if [c,d] is any contiguous interval of $I(Q,\lambda)$, $\omega(F,[c,d]) < M(d-c)$; hence $F \in AC^*$ on $I(Q,\lambda)$. In the case of (C.3) note that if $x \in Q$ then $|F(x+h)-F(x-h)| \leq 2hA$, and so, $\sum \omega_\lambda(K) \leq 4A \sum_K |K| \leq \frac{8A}{2+\lambda}(b-a)$, by (5) of section 2.

The first four operations of T_s-totalisation are just those of the classical T*-totalisation; see Bullen [3]. (C.2) allows us to start the process with (0.1), Lebesgue integration on closed intervals not meeting S([a,b]), (C.1) permits (0.2), evaluating certain limits, and (0.3), finite additivity, requires the use of no conditions. These three operations solve (P.1), see paragraph 4.

From the discussion in paragraph 5; it suffices to consider (P.2) for perfect sets of definite type.

If the total is known on the closed contiguous intervals of an $I(Q,\lambda)$ (= Q if $P \in D_1$ and λ is large enough), then by (C.2) if $x,y \in I(Q,\lambda)$, $F(y)-F(x)$ is given by (0.4), $F(y)-F(x) = \int_x^y f\, 1_{I(Q,\lambda)} + (x \sum y)T_s - \int_{c_n}^{d_n} f$,

where $[c_n,d_n]$, $n \geq 1$ denote the closed contiguous intervals of $I(Q,\lambda)$. This solves (P.2) when $P \in D_1$.

If $P \in D_2$ let $c \leq x_o < x_1 < \dots < x_{4m+2} \leq d$ be as in paragraph 7 except that x_{4n-2}, x_{4n+1} are defined as satisfying:

$$\frac{x_{4n-2}+x_{4n+1}}{2} \in P \cap [x_{4n-1},x_{4n}] = P \cap K_n,\ 2|K_n| \leq x_{4n+1}-x_{4n-2} < 4|K_n|;$$

if $\lambda > 8$ a sequence defined this way is increasing as required. Using the suffixes 1 and 2 as in paragraph 7.

$$F(d)-F(c) = \sum_1 \{F(x_{k+1})-F(x_k)\} + \sum_2 \{F(x_{k+1})-F(x_k)\};$$

by (C.2) $\lim_{\lambda\to\infty} \sum_2 = 0$ and this justifies the following as (0.5) of

T_s-totalisation:

$$F(d)-F(c) = \lim_{\lambda\to\infty} \sum_1 \{F(x_{k+1})-F(x_k)\}.$$

Then by (C.1), and (0.3), suffices to solve (P.2) in this case.

Suppose now that $P \in D_3$ then, as in paragraph 8, it suffices to calculate $F(y)-F(x)$, $y,x \in \Omega(P)$, and to obtain an approximation for y, $x \in I(P,\lambda)$ provided the error tends to zero as $\lambda \to \infty$. Let $P = I(P,\lambda) \cup J(P,\lambda)$ and let $[c,d]$ be a closed contiguous interval of $I(P,\lambda)$ in which the λ-maximal portions of $J(P,\lambda)$ accumulate at both c and d, (it suffices to consider this case, see paragraph 8.) If $c < x < x' < d$, x, $x' \in J(P,\lambda)$, then proceeding as we did above in the case of $P \in D_2$, (see also paragraph 8), we can obtain:

$$F(x')-F(x) = (x \textstyle\sum_1 x')\{F(x_{k+1})-F(x_k)\} + (x \textstyle\sum_2 x')\{F(x_{k+1})-F(x_k)\};$$

by (C.2) $\lim\limits_{\substack{x\to c\\ x'\to d}} \sum_2$ exists and is just $(c \sum_2 d)\{F(x_{k+1})-F(x_k)\}$, and by (C.4) this last expression tends to zero or $\lambda \to \infty$. Then if $x,y \in \Omega(P)$ we obtain $F(y)-F(x)$ by the following, that is (0.6) of this process:

$$\lim_{\lambda\to\infty} \{ \int_x^y f\, 1_{I(P,\lambda)} + (x \textstyle\sum y) \lim\limits_{\substack{x_n\to c_n\\ x_n'\to d_n}} \textstyle\sum_{1,n}\}$$

where of course $\sum_{1,n}$ is the $\sum_1$ for the closed contiguous interval $[c_n,d_n]$, of $I(P,\lambda)$, $n \geq 1$. This completes the discussion of T_s-totalisation.

10. Peano Derivatives Relative to a Set. The results in this section can be found in Denjoy [10].

Definition 3. (The notation is that of section 1 paragraph 3). The index of a set P is said to be uniformly bounded by A iff $\exists h > 0$ such that $\forall x \in P \; \exists x + h_o \in P$, $|h_o| > h$, and $\alpha(P,x,h_o) < A$.

Remark (30). If at all points of P the index is finite then the set of points of P at which the index is not uniformly bounded is nowhere dense in P; [10,p.293].

Definition 4. Let $F: H \to R$, $x \in H \cap H'$, if $\exists \alpha_1, \ldots, \alpha_n$, not depending on h, such that if $x + h \in H$

$$F(x+h) = F(x) + \sum_{k=1}^{n} \alpha_k \frac{h^k}{k!} + o(h^n), \text{ (as } h \to 0),$$

then α_k is called the kth-Peano derivative of F relative to H, at x, written $F^H_{(k)}(x)$, $1 \le k \le n$, if $F^H_{(k)}(x)$ exists, $1 \le k \le n-1$, define $\gamma_n(F,x,h,H)$ by,

$$\frac{h^n}{n!} \gamma_n(F,x,h,H) = F(x+h) - F(x) - \sum_{k=1}^{n-1} F^H_{(k)}(x) \frac{h^k}{k!},$$

($x \in H \cap H'$, $x+h \in H$), then the upper (lower), nth-Peano derivative of F relative to H, at x is

$\limsup_{h\to 0} \gamma_n(F,x,h,H)$ ($\liminf_{h\to 0} \gamma_n(F,x,h,H)$) written $\overline{F}^H_{(n)}(x)$ ($\underline{F}^H_{(n)}(x)$).

Theorem 1. Let $F^P_{(n)}$ exist at all points of a perfect set P of uniformly bounded index; let $Q \neq \phi$, be a perfect subset of P such that $\exists$ α, η for which if $|h| < \eta$, then $|h\, \gamma_{n+1}(F,x,h,P)| < \alpha$, then;

(a) $F^P_{(n)}$ is bounded on Q:

(b) $\exists$ μ such that $\forall x \in Q$ the oscillation of $F^P_{(n)}$ on Q, at x, is less than $\mu\alpha$.

(c) $F^P_{(n-1)}$ is continuous relative to Q, on Q and has finite Dini-derivates relative to Q, at each point of Q, that differ by less than $2\mu\alpha$;

(d) if $1 \leq k \leq n-2$, then $F^P_{(k)}$ is continuous relative to Q, on Q, and $\forall x \in Q, (F^P_{(k)})^Q_{(n-k-1)}(x)$ exists, and further $(F^P_{(k)})^Q_{(m)}(x) = F^P_{(k+m)}(x)$, $1 \leq m \leq n-k-1$.

Remark (31). This remarkable result depends on the very elementary but fundamental Theorem 1 of Denjoy [10].

(32). The above result is certainly false if P has an infinite index at each point. Let $\lambda_n \geq 0$, $\lambda_1 = 1$, $\lambda_\infty = 0$, $\frac{\lambda_{n+1}}{\lambda_n} < \frac{1}{8}$, $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 0$. Let P be the perfect set, having infinite index at each point, defined by

$$P = \{x : x = \sum_{k=0}^{\infty} (-1)^k \lambda_{i_k}, \text{ for some } i_k, 0 \leq i_o \leq i_1 \ldots\}.$$

Define $F : P \to R$ by $F(x) = \sum_{n=1}^{\infty} (-1)^n \frac{\lambda^3_{i_n}}{6}$; then F is continuous on P, $F^P_{(1)} = F^P_{(2)} = 0$, $F^P_{(3)} = 1$; so $F^P_{(3)} \neq (F^P_{(1)})^P_{(2)}$ and $F^P_{(3)} \neq (F^P_{(2)})^P_{(1)}$.

(33). If $F: P \to R$ is continuous then $F^P_{(n)}$ is Baire-1 on P; Denjoy [10; Theorem IV].

Definition 5. Let $F: H \to R$, $P \neq \phi$, a perfect subset of H, if $x \in P$ then F is regular relative to P (of order n) at x iff:

(a) $\forall x \in P$, $F^P_{(n-1)}(x)$ exists with $\overline{F}^P_{(n)}(x)$, $\underline{F}^P_{(x)}(x)$ finite;

(b) if $1 \leq k \leq n-1$ then $\forall x \in P$, $(F^P_{(k)})^P_{(m)}(x) = F^P_{(k+m)}(x)$, $1 \leq m \leq n-k-1$, and $\overline{(F^P_{(k)})}^P_{(n-k)}(x)$ an $\underline{(F^P_{(k)})}^P_{(n-k)}(x)$, are finite.

Theorem 2. Let H be a set with index finite at each point, and with $\overline{F}^{H}_{(n)}(x)$, $\underline{F}^{H}_{(n)}(x)$ finite, $x \in H$, then $\forall\, P \neq \phi$, perfect subset of H, the set of points of P where F is not regular relative to P is nowhere dense in P.

11. Some Problems.

(1) If $f \in N^2_{s,g}(a,b)$ then f is James P^2-integrable and $P^2\text{-}\int_{(a,b)}^{x} f = N^2_s\text{-}\int_{(a,b)}^{x} f$, $a \le x \le s$. What is the relationship between the P^2-integral and the T^2_s-total ? (See James [1]).

(2) If $f \in N_{s,g}$ then f is James symmetric P^1-integrable and the integral are equal. What is the relationship between the symmetric P^1-integral and T_s-totalization ? (See James [2], Mukhopadhyay).

(3) Denjoy has generalized his T^2_s-totalization by allowing certain approximate notions into (C.1)-(C.5); thus (C.1) is replaced by approximate continuity, (C.2) by approximate smoothness; see Denjoy [11;pp.465-481]. Skvorcov, [1], has shown that this generalization does not integrate all finite second order symmetric approximate derivatives of approximately continuous functions. How can Denjoy's totalization be changed so as to integrate all such derivatives ? Skvorcov [1], has modified the James P^2-integral to obtain is Perron integral that integrates these derivatives. How is it related to the Denjoy modification ?

(4) Bhattacharrya has defined a Perron integral that integrates all finite approximate symmetric derivatives of approximately continuous

functions. Modify the T_s-totalisation so as to obtain a process that will integrate such derivatives and relate it to this Perron integral, and to the James symmetric P^1-integral.

(5) Can the results of section 10 be extended to the higher order symmetric derivatives of de la Vallée Poussin ?

(6) In [10] Denjoy defined an nth-order total. Define an equivalent Perron integral; (such an integral will be given in a forthcoming paper of the present author.)

(7) Is it possible to make the T_s^2 and T_s-totalizations completely constructive ?

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