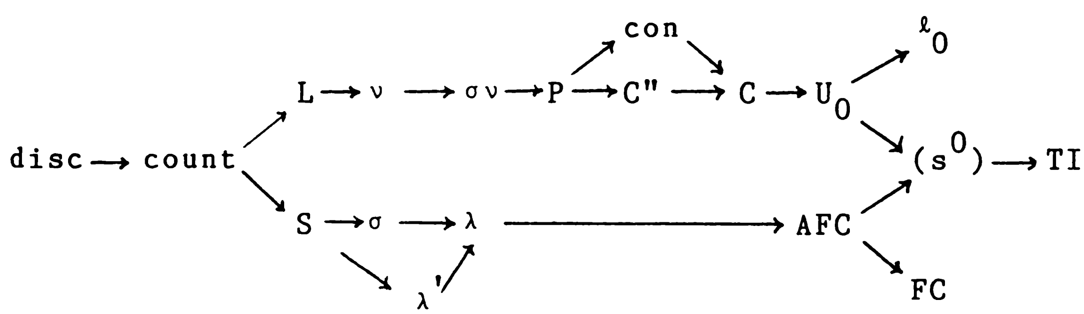
Jack B. Brown, Dept. of Mathematics, Auburn University, AL 36849

SINGULAR SETS AND BAIRE ORDER

I. Singular Sets.

We are interested in separable metric spaces X which are of countable Baire Order (defined below) and their relationship with those spaces which have certain "singularity properties" such as those discussed in Sec. 40 of Kuratowski's **Topology Vol. 1** and more recently in the expository articles [BrCo82] and [Mi84]. In particular, we will be interested in those properties included in the following diagram of implications (we assume X is a subspace of the reals R):

(I) 

The AFC ⟶ FC implication requires that X have no isolated points. The properties are defined as follows: "disc" = discrete, "count" = countable, "L" = Lusin (i.e. every nowhere dense in R set intersects X in a countable set), "ν" = every nowhere dense in X subset of X is countable, "σν" = countable union of ν spaces, "con" = concentrated about a countable subset Y of R (i.e. every open set containing Y contains all but countably many points of X, "P" = concentrated about a countable subset of X, " C" " = for every system $\{U(x,n) \mid x \in X \text{ and } n=1,2,...\}$ of open sets such that

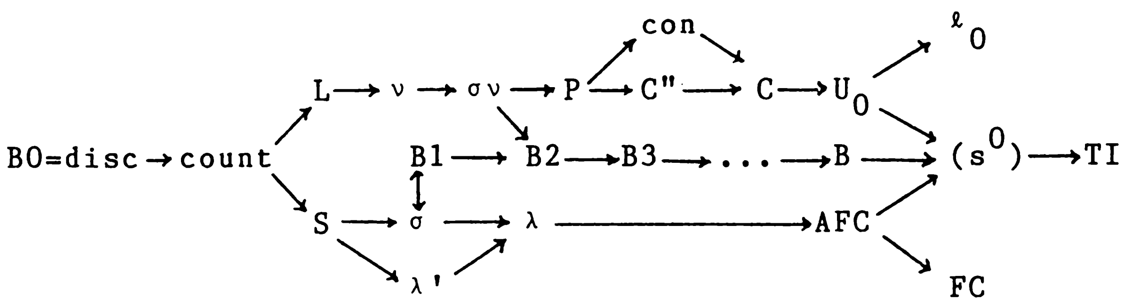
$x \in U(x,n)$ for every x and n, there exists a sequence $x_1, x_2, \ldots$ of elements of X such that $X \subseteq U(x_1,1) \cup U(x_2,2) \cup \ldots$, " C " = strong measure zero (i.e. for every sequence $t_1, t_2, \ldots$ of positive numbers, there exists a sequence $x_1, x_2, \ldots$ of elements of X such that $X \subseteq N(x_1,t_1) \cup N(x_2,t_2) \cup \ldots$, where $N(x,t)$ denotes the t-neighborhood of x, "U_0" = universal null (i.e. of measure zero with respect to the completion of every continuous Borel measure on R), "ℓ_0" = of Lebesgue measure zero, "S" = Sierpinski (i.e. every ℓ_0 subset of X is countable), "σ" = every relative F_σ subset of X is a relative G_δ, "λ" = rarified (i.e. every countable subset of X is a G_δ relative to X), "λ'" = the union of X and any countable subset of R still has property λ, "FC" = first category, "AFC" = always first category (i.e. for every perfect subset Y of R, $X \cap Y$ is first category relative to Y), "(s^0)" = Marczewski null (i.e. every perfect subset Y of R contains a perfect subset Z such that $X \cap Z = \phi$), "TI" = totally imperfect (i.e. X contains no perfect subset).

II. Countable Baire Order.

Let $G_0, G_1, \ldots, G_\alpha, G_{\alpha+1}, \ldots$ denote the usual transfinite sequence with union the class of relative Borel subsets of X, where G_0 denotes the relative open subsets of X, G_1 the relative G_δ sets, G_2 the relative $G_{\delta\sigma}$ sets, etc. The space X is said to be of "Baire (or Borel) Order α" if α is the first ordinal for which $G_\alpha = G_{\alpha+1}$. We denote this α by "ord(X)". If α is a countable ordinal, we denote the property (or class) of spaces X for which $\text{ord}(X) \leq \alpha$ by "Bα", and "B" will just denote the property of having countable Baire order. The question of whether there exist spaces of every countable order was raised by Banach and Mazurkiewicz. It would appear from reading the papers [Sz30] and [Po30] that Banach had conjectured that the answer was "no" and that Mazurkiewicz had conjectured that the answer was "yes". It is easy to see that B0 = disc and that B1 = σ. It was shown in [Si30] that count $\longrightarrow$ B1, in [Sz30] that uncountable S spaces have order = 1 (so S $\longrightarrow$ B1), and in [Po30] that uncountable L spaces have order = 2 (so L $\longrightarrow$ B2). This last result was extended from property L to property $\sigma\nu$ in [Br77]. Since it follows from CH that there are uncountable S spaces and uncountable L spaces, we had a positive CH-solution to the Baire Order Problem for α = 0, 1, and 2 in 1930. The problem remained essentially at that stage until 1979 when Miller and Kunen (see [Mi79]) showed under CH that $B_\alpha \nrightarrow B_\beta$ for every countable $\alpha < \beta$, and Miller showed in [Mi79] that it is also consistent that B1 = Bα for every countable ordinal $\alpha > 1$.

Theorem: B $\longrightarrow$ (s^0).

This theorem, together with the above stated results, shows that the countable Baire Order properties fit properly with the singularity properties of (I) as follows:

(II) 

III. Some open problems.

First, we ask if there might be some improvements possible in the implications which are indicated in (II). Can we improve on the implications going from the upper row to the middle row? It was shown in [Br77] that under CH, $P \not\to B2$ and it was shown in [Mi79] that it is consistent that $P \not\to B$. **(P1):** Is it the case that under CH, $P \not\to B$? Can we establish implications going from the middle row to the upper or lower rows? It was naively conjectured in [Br77] that $B2 \longrightarrow \sigma\nu$ might be the case, but it was pointed out in [Ga78] that if (under CH) X is the union of an uncountable L subspace of [0,1] and an uncountable S subspace of [1,2], then X satisfies B2 but is neither FC nor ℓ_0. It has been shown in [FlMi80] that it is consistent that there be a B1 set which is con. Since con and λ' are incompatible, it follows that it is consistent that $B1 \not\to \lambda'$, but **(P2):** we do not know whether $B1 \not\to \lambda'$ is the case under CH. Finally, consider the implications which go from the bottom row to the middle row. It was shown in [MzSz37], using an dimension argument, that under CH it is the case that $\lambda \not\to B1$. That dimension argument does not carry over to show $\lambda' \not\to B1$, but an alternative dimension argument is given in [Wa84] to show this. The result of [MzSz37] was drastically improved in [Mn77], where it was shown that under CH, $\lambda \not\to B$. **(P3):** Can it be shown under CH that $\lambda' \not\to B$?

It was shown in Sec. IX of [BrCo82] that certain properties in the upper row of (I) were incompatible with certain properties in the lower row, but that other compatibilities were possible. For example, it was shown that "L^FC is not possible" (by this we mean that there is no uncountable set which is both L and FC) but under CH, ν^FC is possible. $\sigma\nu$^AFC is not possible but under CH, P^AFC is possible [FrTa80]. **(P4):** Is (ord(X)=2)^AFC possible under CH? Concerning (P4), we note that it was shown in [Mi84] that it is consistent that (ord(X)=α)^λ be possible for every α. P^λ is not possible, but under CH, con^λ (or even con^B1 [Mi84]) is possible. **(P5):** It is unknown whether C"^λ is possible under CH. con^λ' is not possible, but U_0^λ' is possible (even in ZFC [Si45]). **(P6):** it is unknown whether C^λ' is possible under CH (this will not be answered in ZFC because it has been shown in [La76] that C$\longrightarrow$ count, called Borel's Conjecture, is consistent).

Finally, we state a general problem. **(P7):** Determine the combinatorial properties of the Bα spaces. In other words, investigate whether the properties Bα are preserved (1) in subspaces, (2) in finite or countable intersections, unions, or products, or (3) under continuous, measurable, or other types of transformations. For example, it was shown in [BrGa79] that the increasing countable union of B2 spaces is B3, but this result can probably be improved. There will be CH counterexample to certain conjectures which might be necessary, and the techniques of Miller and Kunen might make construction of these examples more tractable now.

REFERENCES

[Br77] J. B. Brown, On the Baire order of concentrated spaces and L_1 spaces, Fund. Math. 94 (1977), 153-160.

[BrCo82] J. B. Brown and G. V. Cox, Classical theory of totally imperfect spaces, Real Analysis Exchange 7 (1982), 185,232.

[BrGa79] J. B. Brown and R. J. Gardner, Generalized Lusin sets and the Baire order problem, Bull. d'Acad. Polon. 27 (1979), 745-750.

[FlMi80] W. G. Fleissner and A. W. Miller, On Q-sets, Proc. Amer. Math. Soc. 78 (1980), 280-284.

[FrTa80] H. Friedman and M. Talagrand, Un ensemble singular, Bull. Sci. Math. France 104 (1980), 337-340.

[Ga78] R. J. Gardner, personal communication (1978).

[La76] R. Laver, On the consistency of Borel's conjecture, Acta Math. 137 (1976), 151-169.

[Mi79] A. W. Miller, On generating the category algebra, and the Baire order problem, Bull. d'Acad. Pol. 27 (1979), 751-755.

[Mi84] A. W. Miller, Special subsets of the real line, in Handbook of Set Theoretic Topology, North Holland 1984, K. Kunen and J. Vaughn, Editors.

[Mn77] R. D. Mauldin, On rectangles and countably generated families, Fund. Math. 95 (1977), 129-139.

[MzSz37] S. Mazurkiewicz and E. Spilrajn-Marczewski, Sur la dimension de certains ensembles singulars, Fund. Math. 28 (1937), 305-308.

[Po30] G. Poprougenko, Sur un problem de M. Mazurkiewicz, Fund. Math. 15 (1930), 195-198.

[Si30] W. Sierpinski, Sur les images de Baire des ensembles linears, Fund. Math. 15 (1930), 284-286.

[Si45] W. Sierpinski, Sur deux consequences d'un theoreme de Hausdorff, Fund. Math. 33 (1945), 269-272.

[Sz30] E. Spilrajn-Marczewski, Sur un probleme de M. Banach, Fund. Math. 15 (1930), 212-214.

[Wa84] J. T. Walsh, Marczewski sets, measure, and the Baire property, doctoral dissertation, Auburn University, Alabama, 1984.