Real Analysis Exchange Vol. 10 (1984-85)

K.M. Garg, Department of Mathematics
University of Alberta, Edmonton, Alberta, Canada T6G 2G1

DERIVATIVES OF VARIATION FUNCTIONS AND OF MUTUTLLY SINGULAR AND RELATIVELY ABSOLUTELY CONTINUOUS FUNCTIONS

Introduction. We present here a theorem dealing with the relations between the derivative of a function of bounded variation and those of its positive, negative and total variation functions. Further, given two functions of bounded variation f and g on an interval I, we define the notions of mutual singularity of f and g and of absolute continuity of f relative to g similar to those of measures and then present characterizations of these two properties in terms of the derivatives of f and g.

Let $\mathcal{B}$ denote the space of all real valued functions of bounded variation on the interval $I \equiv [a,b]$. For each $f \in \mathcal{B}$ we use f^+, f^- and $\overline{f}$ to denote the positive, negative and the total variation functions of f on I. Further, for each set $E \subset \mathbb{R}$, $|E|$ is used to denote the Lebesgue outer measure of E.

1. Derivatives of variation functions

Given a function $f \in \mathcal{B}$, let C denote the set of points where f is continuous and μ be the outer measure generated by $\overline{f}$ on I (see [3, pp. 64,96] for definitions). It has been proved by de la Vallee Poussin [3, p. 127] that there exists a subset A of C such that $|A| = \mu(A) = 0$ and such that $\overline{f}'(x) = |f'(x)|$ for each $x \in C \sim A$. Following in a refined version of this theorem which was obtained in [1].

THEOREM 1. Given $f \in \mathcal{B}$, there exists a decomposition $I = A \cup B \cup C \cup D$ such that the following holds:

(a) $|A| = |f(A)| = |f^{\pm}(A)| = |\overline{f}(A)| = 0$.

(b) At each point $x \in B$, f, $f^{\pm}$ and $\overline{f}$ have finite derivatives and

(i) if $f'(x) \geq 0$, then $\overline{f}'(x) = f^{+'}(x) = f'(x)$ and $f^{-'}(x) = 0$.

(ii) if $f'(x) < 0$, then $\overline{f}'(x) = f^{-'}(x) = -f'(x)$ and $f^{+'}(x) = 0$.

(c) If $x \in C$, then $\overline{f}'(x) = f^{+'}(x) = f'(x) = \infty$ and $|f^{-}(C)| = |C| = 0$.

(d) If $x \in D$, then $\overline{f}'(x) = f^{-'}(x) = -f'(x) = \infty$ and $|f^{+}(D)| = |D| = 0$.

It was shown further in [1] that nothing can be said in general about the derivability or the derivative of f^{-} at the points of C . The same holds for f^{+} at the points of D .

2. Mutually singular functions

Let two functions $f, \ g \in \mathcal{B}$ be called *mutually singular*, or $f \perp g$, if $\overline{f} = (\overline{f}-\overline{g})^{+}$ and $\overline{g} = (\overline{f}-\overline{g})^{-}$. Following is an equivalent analytical definition of mutual singularity. We use here Vf to denote the total variation of f on I .

THEOREM 2. Two functions f , $g \in \mathcal{B}$ are mutually singular iff for every $\epsilon > 0$ there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ of I with a decomposition $\{1,2,\ldots,n\} = J \cup K$ such that

$$\sum_{i \in J} |f(x_i) - f(x_{i-1})| > Vf - \epsilon \quad \text{and} \quad \sum_{i \in K} |g(x_i) - g(x_{i-1})| > Vg - \epsilon .$$

The following characterization of mututal singularity is obtained with the help of Theorem 1.

THEOREM 3. Two functions f , $g \in \mathcal{B}$ are mutually singular iff the following conditions hold:

(a) f and g are not discontinuous from the same side at any point of I .

(b) $f'(x)\, g'(x) = 0$ for almost every x in I and

(c) the set E of points where both f and g have infinite derivatives has a decomposition $E = A \cup B$ such that $|\overline{f}(A)| = |\overline{g}(B)| = 0$.

The conditions (a), (b) and (c) of the above theorem become redundant when any of the functions f and g in continuous, singular or it satisfies the Lusin's condition (N) respectively. Thus $f \perp g$ whenever f is absolutely continuous and g is singular. Further, f is singular iff $f \perp g$ where g is the identity function on I.

H. Kober [2] defined two nondecreasing functions f and g on [0,1] to be "contravariants" if $f = (f-g)^+$ and $g = (f-g)^-$, i.e. if $f \perp g$ and $f(0) = g(0) = 0$. A characterization of contravariance follows thus from Theorem 3 which is similar to it.

3. Relative absolute continuity

Given $f, g \in \mathcal{B}$, we call f *absolutely continuous relative* to g, or $f \ll g$, if, for each function $h \in \mathcal{B}$, if $h \perp g$ then $h \perp f$. Following is an equivalent analytical definition:

THEOREM 4. Given $f, g \in \mathcal{B}$, $f \ll g$ iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that, for each finite set of nonoverlapping intervals $\{[a_i,b_i] : i = 1,2,\dots,n\}$ in I, if $\sum_{i=1}^{n} \{\bar{g}(b_i) - \bar{\delta}(a_i)\} < \delta$ then $\sum_{i=1}^{n} \{\bar{f}(b_i) - \bar{f}(a_i)\} < \epsilon$.

Thus f in absolutely continuous iff it is so relative to the identify function on I. The following characterization of relative absolute continuity is obtained with the help of Theorem 3.

THEOREM 5. Given $f, g \in \mathcal{B}$, $f \ll g$ iff the following conditions hold:

(a) f is continuous at each point $x \in I$ from the side from which g is continuous,

(b) $f'(x) = 0$ for almost every x for which $g'(x) = 0$ and

(c) if A and B are the sets of points where f and g have infinite derivatives respectively, then $|\bar{f}(A \sim B)| = 0$ and, for each set $E \subset A \cap B$, if $|\bar{g}(E)| = 0$ then $|\bar{f}(E)| = 0$.

The conditions (a), (b) and (c) of the above theorem become redundant, as before, when f is continuous, singular or it satisfies the condition (N) respectively. Further, if $f \ll g$, then f is by this theorem continuous, singular or absolutely continuous whenever g is so.

Remark. The above results are found useful in the construction of functions of bounded variation with complicated properties, e.g. a continuous singular function which is not monotone on any set A such that $|A| > 0$.

REFERENCES

[1] K.M. Garg, On derivatives of variation functions, The Eoin L. Whitney Memorial Collection, Mathematical papers contributed by friends, colleagues and pupils, University of Alberta, Edmonton 1967, pp. 79-95.

[2] H. Kober, On decompositions and transformations of functions of bounded variation, Ann. Math. 53(1951), pp. 565-580.

[3] S. Saks, Theory of the Integral, Monografie Matematyczne 7, Warsaw - Lwow, 1937.