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# Uncountable-order sets for radial-limit functions

The content of this lecture is contained in the paper with the same title co-authored by Robert Berman, Togo Nishiura and George Piranian [2]. We begin with some notation.

Let $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ and $C = \{z \in \mathbb{C} \mid |z| = 1\}$, where $\mathbb{C}$ is the set of complex numbers. The closure of $\Delta$ is then $\bar{\Delta} = \Delta \cup C$. We let $\check{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane. For a function $f : \Delta \to \mathbb{C}$ we define the radial limit function $f^* : C \to \check{\mathbb{C}}$ by

$$f^*(\eta) = \lim_{r \to 1} f(r\eta) \quad , \eta \in C,$$

whenever the limit exists. An <u>inner function</u> is a bounded analytic function such that $|f^*(\eta)| = 1$ for almost all $\eta \in C$. An example of an inner function is a <u>Blaschke product</u>

$$B(z) = z^m \prod_n \frac{\bar{a}_n}{|a_n|} \left(\frac{z-a_n}{1-\bar{a}_n z}\right) ,$$

$a_n \in \Delta$ , $\sum_n (1-|a_n|) < \infty$.

By a <u>point of uncountable order</u> of a function $g : X \to Y$, we mean a point $y$ of $Y$ such that $g^{-1}(y)$ is an uncountable set. The set of points of uncountable order of $g$ will be denoted by $U(g)$.

In their paper [3], MacLane and Ryan proved the following theorem on Blaschke products by means of Riemann surface arguments.

Theorem [3]. Corresponding to each closed set W on the unit circle C there exists a Blaschke product B such that $U(B^*) = W$ and $B^*(\eta)$ has modulus 1 whenever it exists.

MacLane and Ryan asked whether it is feasible to prove the theorem by describing suitable Blaschke products in terms of their zeros.

The above question is resolved in the affirmative. We first construct a continuous function $g : P \to C$, where P is a suitably constructed perfect subset of C, such that $U(g) = W$ and then describe the zeros of a Blaschke product so that $U(B^*) = U(g)$. In other words, the question is in reality a Real Analysis problem. Our methods establish the following theorems.

Theorem. If W is an analytic subset of C, there exists a Blaschke product B such that $U(B^*) = W$ and $|B^*(\eta)| = 1$ for each $\eta \in C$. Moreover, if B is a Blaschke product then $U(B^*)$ is an analytic set.

Theorem. Every uncountable analytic subset of C contains a perfect set P such that if a subset H of P is of the type $G_{\delta\sigma}$, then some Blaschke product has a radial limit at each point of $C\backslash H$ and at no point of H.

These two theorems are consequences of two elementary principles about Blaschke products which will be stated next.

Moderation Lemma. (Berman and Piranian). If $z \in \Delta$ and $B(z) = \prod_n \frac{\bar{a}_n}{|a_n|}\left(\frac{z-a_n}{1-\bar{a}_n z}\right)$ is a Blaschke product with $a_n \neq z$ then $|B(z)-1| \leq M \sum_n \frac{1-|a_n|}{|a_n - z|}$, where M is a universal constant.

Angular Adjustment Lemma. (Berman and Piranian). Let $\eta \in C$ and $I(\eta)$ and $J(\eta)$ be subarcs of $C$ with center $\eta$ and lengths $\delta$ and $\gamma$. Denote by $C(I(\eta))$ the part of the circle orthogonal to $C$ at the end-points of $I(\eta)$ which is interior to $\bar{\Delta}$. Also, denote by $d(J(\eta))$ the part of the disk interior to $C(J(\eta))$ and $\bar{\Delta}$. Let $b(a,z) = \frac{\bar{a}}{|a|}\left(\frac{z-a}{1-\bar{a}z}\right)$. Then there is a universal constant $K$ such that, for $a \in C(I(\eta))$ and $z \in d(J(\eta))$,

$$|b(a,z) - \exp\left[2i \arg \left(\frac{a-\eta}{i\eta}\right)\right]| < K \frac{\gamma}{\delta}$$

when $0 < 2\gamma < \delta < \frac{\pi}{2}$.

Finally, with the aid of complex analysis results of Rudin [5] concerning the Banach algebra of continuous functions on $\bar{\Delta}$ which are analytic on $\Delta$ and the Arakeljan Approximation Theorem [1], we are able to prove by our Real Analysis methods the following theorems.

Theorem. A set $W$ of the extended complex plane is the set $U(f^*)$ for some (bounded) analytic function $f$ on $\Delta$ if and only if it is a (bounded) analytic set.

Theorem. If $H$ is a nowhere-dense set of the type $G_{\delta\sigma}$ on $C$, there exists an analytic function $f$ on $\Delta$ such that $H$ is the set of points $\eta$ on $C$ for which $f^*(\eta)$ does not exist.

Question. It is known that some inner function $f$ has $U(f^*) = \bar{\Delta}$ [4]. Can one show that for any analytic set $W$ of $\bar{\Delta}$ there is an inner function $f$ such that $U(f^*) = W$?

References


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