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APPROXIMATE SMOOTHNESS OF CONTINUOUS FUNCTIONS

In the literature there are numerous constructions of continuous nowhere differentiable real valued functions on the real line R. One which holds particular interest for the present author is the function $f:R \to R$ constructed by L. Filipczak [3] having the property that it has a symmetric derivative nowhere; i.e., at no point does

$$\lim_{h \to 0^+} [f(x+h) - f(x-h)]/2h$$

exist. More specifically, at each $x \in R$ this function satisfies both

$$(1) \quad \lim_{h \to 0^+} \sup[f(x+h) - f(x-h)]/2h = +\infty$$

and

$$(2) \quad \lim_{h \to 0^+} \inf[f(x+h) - f(x-h)]/2h = -\infty.$$

P. Kostyrko [4] subsequently showed that the class of all functions satisfying both (1) and (2) at each $x \in [0,1]$ is residual in the metric space $C([0,1])$ of continuous functions on [0,1] with the supremum metric. The present author then noted [1] that the methods of Filipczak and Kostyrko could be used to show that the class of functions in C([0,1]) satisfying everywhere the two conditions

$$(1^*)\quad \operatorname{ap}\limsup_{h\to 0^+}[f(x + h) - f(x - h)]/2h = +\infty$$

and

$$(2^*)\quad \operatorname{ap}\liminf_{h\to 0^+}[f(x+h) - f(x-h)]/2h = -\infty$$

is also residual, where ap lim sup (inf) denotes the approximate limit superior (inferior).

An alternate, or in a certain sense a complementary, symmetry criterion for a function at a point x is that of approximate smoothness, where f is said to be approximately smooth at x if

$$\operatorname{ap}\lim_{h\to 0^+}[f(x+h) + f(x-h) - 2f(x)]/h = 0.$$

It is clear that a function possesses a finite approximate derivative at a point x if and only if it is both approximately smooth at x and possesses a finite approximate symmetric derivative at x. It therefore seems of some interest to inquire whether a continuous function can fail to satisfy either of these symmetry criteria at each point of an interval.

The purpose of the talk having the title above, which was given at the St. Olaf Symposium, was to announce that such functions do exist and, not surprisingly, are residual in $C([0,1])$. More precisely, the collection of functions which satisfy (1^*), (2^*), and

$$(3^*) \quad \text{ap} \limsup_{h \to 0^+} |f(x+h) + f(x-h) - 2f(x)|/h = +\infty$$

at each point is residual in $C([0,1])$. The proof is carried out in [2].

References

1. M. J. Evans, On continuous functions and the approximate symmetric derivative, Colloq. Math. 31(1974), pp. 129-136.

2. M. J. Evans, Approximate smoothness of continuous functions (submitted).

3. L. Filipczak, Exemple d'une fonction continue privée de dérivee symetrique partout, Colloq. Math. 20(1969), pp. 249-253.

4. P. Kostyrko, On the symmetric derivative, Colloq. Math. 25(1972), pp. 265-267.