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Intersections of Continuous Functions with Families of Smooth Functions

This paper is a report of results from joint work with A. M. Bruckner, M. Laczkovich and D. Preiss. The proofs are contained in a paper which has been submitted to TAMS. Some open questions are also included.

It is clear that for $f \in C[0,1]$, f is concave or convex on $[0,1]$ if and only if $\text{card}\{x : f(x) = l(x)\} \leq 2$ for all lines $l(x)$. M. Laczkovich posed the following question: If $f \in C[0,1]$ and $\{f = l\}$ is finite for all lines l, then must there be a subinterval of $[0,1]$ on which f is either concave or convex? An affirmative answer was established, which for reference we state here.

Theorem 1. If $f \in C[0,1]$ and $\{f = l\}$ is finite for all lines l, then there exists a subinterval of $[0,1]$ on which f is either concave or convex.

It is then natural to ask what results one might obtain by considering the cardinality of $\{f = l\}$ for fixed $f \in C[0,1]$ and all lines l. The following were obtained:

Theorem 2. If $f \in C[0,1]$ and $\text{card}\{f = l\} \leq 3$ for all lines l, then $[0,1]$ can be decomposed into 5 subintervals on each of which f is concave or convex.

Somewhat surprisingly, we have

Theorem 3. There exists $f \in C[0,1]$ such that card $\{f = l\} \leq 4$ for all lines l, but $[0,1]$ cannot be decomposed into even countably many subintervals of concavity or convexity of f.

Theorem 4. Let K be the Cantor set. There exists $f \in C[0,1]$ such that card $\{f = l\} \leq 5$ for all lines l but $[0,1]$ cannot be decomposed into even countably many subsets of concavity or convexity of f, and f is neither concave nor convex on any portion of K. In addition, f can be chosen to be increasing and Lipschitz.

Theorem 5. There exists $f \in C[0,1]$ such that $\{f = l\}$ is countable for all lines l, but $[0,1]$ contains no subinterval on which f is concave or convex.

As a consequence of Theorem 4, we see that the direct analogue of Theorem 1 obtained by replacing $f \in C[0,1]$ by $f \in C(K)$ for K perfect, does not hold. In fact, even more is true:

Theorem 6. There exists K^* perfect and $f \in C(K^*)$ such that card $\{f = l\} \leq 2$ for all lines l and yet f is not monotonic on any portion of K^*. In fact, the collection of all such f forms a residual subset of $C(K^*)$.

In view of Theorem 6, it is evident that if we are to obtain any results for $f \in C(K)$ for K perfect, we must strengthen the hypotheses. One way is to consider intersections of the graph of $f \in C(K)$ with larger classes of smooth functions. We remark that it is apparent that if $\lambda(K) = 0$, then there exists $f \in C(K)$ such that $\{f = g\}$ is finite for all g differentiable. To see this, just choose $f \in C(K)$ such that $f' \equiv \infty$. Then if g is differentiable, and x_0 is a point of accumulation of $\{f = g\}$ then $g'(x_0) = \infty$. If $\lambda(K) > 0$, the situation is different:

Theorem 7. If $\lambda(K) > 0$, then for each $f \in C(K)$, there exists $g \in C'$ such that $\{f = g\}$ is uncountable.

Corollary 8. For each $f \in C[0,1]$, there exists $g \in C'$ such that $\{f = g\}$ is uncountable.

One open question is: can Corollary 8 be improved? For example, might we get a twice differentiable function g, or, possibly $g \in C^n$ for some $n > 1$? However, Theorem 7 is the best possible in the following sense:

Theorem 9. Given $\varepsilon > 0$, there exists $K \subset [0,1]$ perfect, such that $\lambda(K) \geq 1 - \varepsilon$ and there exists $f \in C(K)$ such that $\{f = g\}$ is finite for all g twice differentiable.

Theorem 1 can be significantly generalized. We need some definitions and notation.

Definition. If f is defined on a set E, the nth divided difference of f at the distinct points $x_0, x_1, \ldots, x_n \in E$ is defined by

$$V(f, x_0, \ldots, x_n) = \sum_{i=0}^{n} \frac{f(x_i)}{w'(x_i)}$$

where

$$w(x) = \prod_{j=0}^{n} (x - x_j).$$

The function f is said to be n-convex on E if $V(f, x_0, \ldots, x_n) \geq 0$ whenever $x_0, \ldots, x_n$ are distinct elements of E. We say f is n-concave if $-f$ is n-convex. Thus f is 0-convex if $f \geq 0$, f is 1-convex if f is increasing and f is 2-convex if f is convex. If E is an open interval and $n \geq 2$, then f is n-convex on E if and only if $f \in C^{n-2}(E)$ and $f^{(n-2)}$ is convex on E.

We let P_n be the collection of polynomials of degree $\leq n$.

We have the following theorem of Čech [1].

Theorem. For $f \in C[0,1]$, if $\{f = p\}$ is finite for all $p \in P_0$, then f is 1-convex or concave on a subinterval of $[0,1]$.

We may now restate Theorem 1 as:

Theorem. For $f \in C[0,1]$, if $\{f = p\}$ is finite for all $p \in P_1$, then f is 2-convex or concave on a subinterval of $[0,1]$.

The above 2 theorems were shown to be special cases of the following more general and stronger result:

Theorem 10. For $f \in C[0,1]$, if $\{f = p\}$ has no bilateral accumulation point for all $p \in P_n$, then f is n+1-concave or convex on a subinterval of $[0,1]$.

Corollary 11. For $f \in C[0,1]$, if $\{f = p\}$ has no bilateral point of accumulation then there is a subinterval $J \subseteq [0,1]$ such that $f|_J \in C^{n-1}(J)$.

A related theorem is:

Theorem 12. If $f \in C[0,1]$, then either there exists a polynomial p such that $\{f = p\}$ is infinite or there exists $g \in C^\infty$ such that $\{f = g\}$ is uncountable.

In either case, we have

Corollary 13. If $f \in C[0,1]$, then there exists $g \in C^\infty$ such that $\{f = g\}$ is infinite.

Interestingly, we have the following:

Theorem (Zahorski [3] answering a question of Ulam [2]). There exists a function $f \in C[0,1]$ such that $\{f = g\}$ is finite for all g analytic.

An open question is the behavior of such a function f, must it be in C^∞ for example? Or, for how large a set in $C[0,1]$ does Zahorski's result hold?

Another open question relates to Corollary 13: Can the hypothesis that f be continuous be weakened (for example to f Darboux or even arbitrary f)?

As a final open question, we might try to extend the results of Theorems 2 and 3 to n-convexity. We first summarize the known results with a "theorem" in the form of a chart.

"Theorem." For $f \in C[0,1]$, if card $\{f = p\} \leq k$ for all $p \in P_n$, then $[0,1]$ can be decomposed into s subintervals on each of which f is $n+1$ convex or concave according to the following chart.

n	k	s
0	1	1
	2	3
	3	There is no s, even countable
1	2	1
	3	5
	4	There is no s, even countable
n	n+1	1

The open question is now evident: For $n > 1$ and $k > n+1$, what are the correct values of s?

References

[1] E. Čech, Sur les fonctions continues qui prennent chaque leur valeur un nombré fîni de fois, Fund. Math. 17, (1931), 32-39.

[2] S. M. Ulam, A collection of mathematical problems, Interscience Publ., New York, 1960.

[3] Z. Zahorski, Sur l'ensemble des points singuliers d'une fonction d'une variable réele admittant les dérivés de tous les ordres, Fund. Math. 34 (1947), 183-245.