Richard J. O'Malley, Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201

The Second Peano Derivative as a Composite Derivative

One of the interesting unsolved problems concerning Peano derivatives is the lack of a precise description of in what sense an $n+1^{\text{th}}$ Peano derivative can be considered as a derivative; in particular as some form of generalized derivative of the associated n^{th} Peano derivative.

In the theory of approximate differentiation the following is known [4],

If a function $f\colon R \to R$ has an approximate derivative f'_{ap} for all x, then for any fixed perfect set P, there is a portion Q of P such that

$$\lim_{\substack{y\to x\\ y\in Q}} \frac{f(y)-f(x)}{y-x} = f'_{ap}(x), \quad \text{for all } x \in Q$$

In general when a pair of functions u and v satisfy the conclusion above with $u = f$ and $v = f'_{ap}$ we say that v is a composite derivative of u [5].

In all previous cases, it has happened that all properties possessed by approximate derivatives have been found to be common to Peano derivatives. Therefore, it is natural to expect the trend to continue to composite differentiation.

In one sense, this is known to be true. Denjoy [1] has established that if a function $f\colon R \to R$ has an $n + 2^{\text{th}}$ Peano derivative f_{n+2}, $n \geq 1$, then the $n + 1^{\text{th}}$ Peano derivative, f_{n+1}, is the composite derivative of the n^{th} Peano derivative f_n. Thus this particular problem reduces to a determination of whether this $n + 2^{\text{th}}$ condition can be eliminated.

Here, however, our scope will be more restricted. It is based on results in [3]. For the remainder we will confine our discussion to the case where $n = 2$.

Because of the above result for approximate derivatives, it is natural to examine the following situation:

Let $f\colon R\to R$ have a derivative f', and a second Peano derivative f_2. Further let f' possess an approximate derivative $(f')'_{ap}$.

The "nicest" possible situation would be if it were possible to conclude that under such conditions $(f')'_{ap} \equiv f_2$. However, this is not the case, as was shown in [3]. Alternately, once we are aware that these two objects can simultaneously exist and still not be identical, it becomes natural to investigate whether the examples in [3] would be modified in such a way as to provide a counterexample to f_2 being a composite derivatife of f'. A moment's thought reveals that this would require an example where $E = \{x\colon f_2 \neq (f')'_{ap}\}$ is dense in some perfect set.

Since it is known that both f_2 and $(f')'_{ap}$ are Baire class 1, the set E is an F_σ. If an example could be constructed where this set E was uncountable then it would contain a perfect set P and would provide the desired counterexample. This relationship provided the motivation for the current research.

Let's suppose E is dense in a perfect set P. Using the result in [4] and a very useful result in [2] it can be assumed that f' is actually the restriction of a differentiable function g over P, and that $g' = (f')'_{ap}$. Let G be an indefinite integral of g. Let $F = f - G$.

Then the following hold:

a) F' exists $= f' - g$, $F' = 0$ over P,

b) F_2 exists $= f_2 - g'$,

c) $(F')'_{ap}$ exists $= (f')'_{ap} - g'$, $(F')'_{ap} = 0$ over P,

d) $\{x: F_2 \neq (F')'_{ap}\} \cap P = \{x: F_2 \neq 0\} \cap P = E \cap P$.

These conditions provide that

$$\lim_{y \to x} \frac{F(y)-F(x)}{\frac{(y-x)^2}{2}} = F_2(x).$$

So at any point x in E, since $F_2(x) \neq 0$, F has a strict local maxima or minima. Hence we have automatically that E must be countable.

This countability is used in conjunction with the Baire category theorem to find a portion, Q, of P where for all x,y in Q $|F(y)-F(x)| \leq (x-y)^2$. Since E was dense in P, it is dense in Q. However, it can be shown, by a relatively straightforward, though tedious computation that this is a contradiction.

Thus we have, at least for $n = 2$:

<u>Theorem</u>: If $f: R \to R$ has an 2^{nd} Peano derivative and f' has an approximate derivative, then $\{x: (f')'_{ap}(x) \neq f_2\}$ is a scattered set and f_2 is a composite derivative of f'.

<u>Remarks</u>:

1. It should be noted that it is not exactly the existence of $(f')'_{ap}$ which is used in the proof. Whenever f' has any composite derivative v, it turns out that f_2 is a composite derivative of f'.

2. The current proof of the theorem does not seem to extend to $n > 2$.

3. The complete proof of these facts and any further results along these lines will be submitted for publication to Fundamenta Mathematicae in early 1985.

References

[1] A. Denjoy, Sur l'integration des coefficients differentials d'ordre supérieur, Fund. Math. 25 (1935), 273-326.

[2] M. Laczkovich, G. Petruska, Baire 1 functions, approximately continuous functions and derivatives, Acta Math. Acad. Sci. Hungar. 25 (1974), 189-212.

[3] C.M. Lee and R.J. O'Malley, The second approximate derivative and the second approximate Peano derivative, Bull. Inst. Math. Acad. Sinica 3 (1975), 193-197.

[4] R. J. O'Malley, Decomposition of approximate derivatives, P.A.M.S. 69 (1978), 243-247.

[5] R. J. O'Malley and C. E. Weil, Selective, β_1-selective, and Composite Differentiation, Acta Math. Hung. 43 (1984), 31-36.