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Another Note on σ-Porous Sets

§0 Introduction

The notion of porosity has recently played an active role in describing sets which are exceptional in one sense or another. However, the set theoretic properties of porous and σ-porous sets remain largely unexplored, and perhaps rightly so as they seemingly play no important role in the investigations to date. There have been a few inroads made though ([Z], [FH]) and these have proven to be both interesting and disturbing.

On the positive side we know that every porous set is contained in a $G_{\delta\sigma}$ σ-porous set and that every porous set is the countable union of uniformly porous sets. A set E is <u>uniformly porous</u> if there is an $\varepsilon > 0$ such that if $x \varepsilon E$, then the porosity of E at x, devoted $p(E,x)$, exceeds ε. On the disturbing side, for example, it is shown in [FH] that there are porous sets which are contained in no F_σ σ-porous set (indeed in no F_σ measure zero set). This fact taken with the aforementioned uniform porosity result shows that there is a uniformly porous set which is contained in no F_σ σ-porous set. Disturbing.

The discussion to follow was motivated by an investigation of covering properties of porous sets and the results of this note will be used elsewhere to help characterize certain cluster set and derivate behaviour. First a bit of notation. If I is an interval and $r > 0$, we denote by $r*I$ that interval which is concentric with I and whose length is r times the length of I. Now suppose that E is a bounded set with $\inf(E) = a$ and $\sup(E) = b$. Then,

$$(a,b) - cl(E) = \bigcup_{n=1}^{\infty} I_n \quad [cl = closure]$$

where the $\{I_n\}$ are pairwise disjoint open intervals. Now, let $EP(N)$ denote the set of endpoints of the intervals $I_1, I_2, \ldots, I_{N-1}$. Then, if there is an $r > 0$ such that

$$E - EP(N) \subset \bigcup_{n=N}^{\infty} r*I_n$$

for each N then E is unifomly porous and indeed countable unions of such sets characterize σ-porous sets. If there is an $r > 0$ such that for each N there is an $N*$ such that

$$E - EP(N) \subset \bigcup_{n=N}^{N*} r*I_n,$$

then E is called r-globally porous, or globally porous with index r. A set is called globally porous if it is r-globally porous for some index r. The notion of σ-globally porous is defined in the obvious manner. The purpose of this note is to investigate this notion of global porosity and relate it to the parental notions of porosity and uniform porosity. In the first section we prove two elementary but subsequently important propositions concerning globally porous sets. In the last section we construct a perfect porous set which is not σ-globally porous. Disturbing again, perhaps.

§1 Some Positive Results

One might on first glance guess that every subset of a globally porous set is itself globally porous and although such is not the case, it almost is, as the following result details.

PROPOSITION 1. *Let* E *be* r*-globally porous, let* $[a,b]$ *be any closed interval, and let* $\varepsilon > 0$. *Then if* $\{I_k\}$ *is the set of components of* $(a-\varepsilon,b+\varepsilon) - cl(E)$, *then for each* K *there is an* $K^* > K$ *such that* $E\text{-}EP(K) \cap [a,b] \subset \bigcup_{k=K}^{K^*} r^*I_k$.

Proof. Let $A = \inf(E)$, $B = \sup(E)$, and let $\{J_n\}$ be the components of $(A,B) - cl(E)$. For each component I_k of $(A,B)\cap(a-\varepsilon,b+\varepsilon) - cl(E)$ there is a unique $n(k)$ such that $I_k \subset J_{n(k)}$. Further, if this $J_{n(k)}$ contains neither $a-\varepsilon$ nor $b+\varepsilon$, then $I_k = J_{n(k)}$. Let $N_1 = \max\{n(k)):k < K\}$. If $a-\varepsilon \notin E$ ($b+\varepsilon \notin E$) let N_2 (N_3) be that index such that $a-\varepsilon \in J_{N_2}$ ($b+\varepsilon \in J_{N_3}$). Otherwise, let N_2 (N_3) $= 0$. There is an N_4 such that if $n \geq N_4$ then $J_n < \varepsilon/r$, so that if $n \geq N_4$ and $J_n \cap (a-\varepsilon,b+\varepsilon) = \emptyset$, then $r^*J_n \cap [a,b] = \emptyset$. Finally, let $N_5 = \max\{N_i : i=1,2,3,4\}$. As E is globally porous, there is an N_5^{**} such that

$$E\text{-}EP(N_5) \subset \bigcup_{n=N}^{N_5^{**}} r^*J_n.$$

However, if $n \geq N_5$, and $J_n \subset (a-\varepsilon,b+\varepsilon)^C$ then $r^*J_n \cap [a,b] = \emptyset$. Hence if we let

$$K^* = \max\{k : N_5 \leq n(k) \leq N_5^{**}\},$$

then

$$E - EP(K) \cap [a,b] \subset \bigcup_{k=K}^{K^*} r^* I_k,$$

because

$$\bigcup_{k=K}^{K^*} r^* J_{n(k)} \cap [a,b] \subset \bigcup_{k=K}^{K^*} r^* I_k.$$

We now are in a position to consider a property of closed globally porous sets which is quite useful. Specifically:

PROPOSITION 2. *If* E *is globally porous, then* $cl(E)$ *is globally porous*.

Proof. Let r be the index of global porosity of E and let $s > r$. We will show that $cl(E)$ is s-globally porous. Let N be given. As E is globally porous there is an N^* such that

$$E - EP(N) \subset \bigcup_{n=N}^{N^*} r^* I_n.$$

However, $\bigcup_{n=N}^{N^*} r^* I_n$ is an open set with only finitely many components and as such there are at most finitely many points of $cl(E)$ which are not covered by this union. As each such point is necessarily the endpoint of one of the intervals $r^* I_n$, it is contained in the larger interval $s^* I_n$. It follows, then, that

$$cl(E) - EP(N) \subset \bigcup_{n=N}^{N^*} s^* I_n$$

and as N was arbitrary the proposition is proved. The proof of the next result is immediate, then from PROPOSITION 2.

COROLLARY 3. E *is σ-globally porous, iff* E *is contained in an* F_σ *σ-globally porous set.*

In light of COROLLARY 3, one can immediately make two remarks. The first is that the example given in [FH] of a σ-porous set which is contained in no F_σ σ-porous set also provides an example of a σ-porous set which is not σ-globally porous. The second is that every σ-globally porous set is contained in an F_σ σ-porous set. The converse of the latter result is also false, and in fact, there are perfect porous sets which are not σ-globally porous. This is the main charge of §2.

§2. An Example

It is easy to see that the notions of globally porous and uniformly porous are distinct notions. In particular, note that the set $S = \{\pm 1/n : n=1,2,\ldots\}$ is uniformly porous, but not globally porous. For our applications, however, it is not these notions, but the notion of σ-globally porous and σ-uniformly porous which play a critical role, and in this section we show that these too are distinct. We should first remark that σ-uniformly porous sets are exactly the same as σ-porous sets as every porous set is σ-uniformly porous. The purpose of this section, then, is to prove the following theorem.

THEOREM 1. *There is a perfect porous set which is not σ-globally porous.*

Prior to the actual proof we present two preliminary constructions which will be used alternately in the inductive construction of the desired set.

1. The sets $J(G,n)$ and $\mathfrak{J}(G,n)$.

Let $n > 3$ be a natural number, and I be the bounded open interval (a,b). First, let J^o be the open interval of length $(b-a)/(n-1)$ centered at $(a+b)/2$. Then define J^i, $i = \pm 1, \pm 2, \ldots$ inductively as follows.

i. Let $J^{+1} = J^{+1}(I,n)$ $[J^{-1} = J^{-1}(I,n)]$ be the open interval centered at the right [left] endpoint of $2*J^o$ with length chosen as the maximum of all lengths such that $(n-1)*J^{+1}$ $[(n-1)*J^{-1}]$ is contained in I.

ii. If $J^i = J^i(I,n)$ $[J^{-i}]$ has been defined, let J^{i+1} $[J^{-i-1}]$ be the interval centered at the right [left] endpoint of $2*J^i$ $[2*J^{-i}]$ with length chosen as the maximum of all lengths such that:

$$(n-1)*J^{i+1} \subset I \qquad [(n-1)*J^{-i-1} \subset I].$$

We let $J(I,n)$ denote the set of all such intervals and let $\mathfrak{J}(I,n)$ be the union of these intervals. The properties of the set $J(I,n)$ which are important to us are

1. $\bigcup_{i=-\infty}^{+\infty} m*J^i = I$ for $2 \leq m \leq n-1$

2. If $x_i \in J^i$ for $i = 0, \pm 1, \pm 2, \ldots$, then $\lim_{i \to +\infty} x_i = b$ and $\lim_{i \to -\infty} x_i = a$.

Now, if G is any bounded open set, let $J(G,n)$ denote the union of all sets $J(I,n)$ where the union is taken over all components I of G. Let $\mathfrak{J}(G,n)$ denote the union of all the sets $\overset{*}{J}(I,n)$ where again the union is taken over all components I of G. Note that $J(G,n)$ is a collection of disjoint open intervals, while $\overset{*}{J}(G,n)$ is the open set whose components are $J(G,n)$.

2. The sets $\overset{*}{C}(H,n)$ and $\overset{*}{O}(H,n)$.

Let $C(n)$ denote the symmetric Cantor set on $[0,1]$ given by

$$C(n) = \{x \varepsilon [0,1] : x \text{ has only } 0 \text{ or } n-1 \text{ in its } n\text{-nary expansion}\}.$$

If $H = [a,b]$ is any compact interval of R, define

$$C(H,n) = \{a+(b-a)x : x \varepsilon C(n)\}, \text{ and}$$

$$O(H,n) = H - C(H,n).$$

Now, if $\overset{*}{H}$ is any union of mutually exclusive closed intervals, let $\overset{*}{C}(\overset{*}{H},n)$ be the union of the sets $C(H,n)$ where the union is taken over the disjoint closed intervals H of $\overset{*}{H}$, let $\overset{*}{O}(\overset{*}{H},n)$ be the union of the open sets $O(H,n)$ where again the union is taken over all the disjoint closed intervals H of $\overset{*}{H}$.

Two important links between this preliminary construction and the previous one can be illustrated in $[0,1]$ using the symmetric Cantor set $C(n)$. They are

$$(\alpha) \quad (0,1) = \bigcup_{I} n * J^{o}(I,n)$$

$$(\beta) \quad (0,1) - C(n) = \bigcup_{I} \left(\bigcup_{i=-\infty}^{+\infty} (n-1) * J^{i}(I,n) \right)$$

where both unparameterized unions are taken over all components I

of $(0,1) - C(n)$. It therefore follows from (α) that if

$$C(n) \subset E \quad \text{and} \quad E \cap \left(\bigcup_{I} \bigcup_{i=-\infty}^{+\infty} J^i(I,n) \right) = \emptyset$$

then E is porous at every point of $C(n)$ and the porosity at such points is at least $1/n$. Further, if the $J^i(I,n)$ can be used to determine the porosity of E at points of $C(n)$, then it follows from (β) that the porosity cannot exceed $1/(n-1)$ at all points of $C(n)$.

We are now in a position to prove our theorem by constructing the desired compact set E. The construction is actually a construction of the complement of E and is inductive in nature.

Stage 1.

a. Let $\overset{*}{H}_1 = H_1 = [0,1]$, $C_1 = C(H_1,4)$, $O_1 = O(H_1,4)$.

b. Let $J_1 = \overset{*}{J}(O_1,4)$.

Note that J_1 is open and $C_1 \subset [0,1] - J_1$. Let

$$\overset{*}{H}_2 = [0,1] - (J_1 \cup C_1)$$

which is, of course, a set of mutually exclusive closed intervals. Note that if $x \in C_1 - \{0,1\}$, then $x \in 4*J_1 - 3*J_1$ while if $x \in \overset{*}{H}_2$, then $x \in 2*J_1$.

Stage 2.

a. Let $C_2 = \overset{*}{C}(\overset{*}{H}_2,5)$ and $O_2 = \overset{*}{O}(\overset{*}{H}_2,5)$.

b. Let $J_2 = \overset{*}{J}(O_2,5)$.

Subsequently, we define $\overset{*}{H}_3 = [0,1] - [(J_1 \cup J_2) \cup (C_1 \cup C_2)]$. The induction then proceeds as in Stage 2 with all indices raised by one. The complement of E is then $\bigcup_{n=1}^{\infty} J_n$ while E itself can be expressed as

$$E = (\bigcup_{n=1}^{\infty} C_n) \cup (\bigcap_{n=1}^{\infty} \overset{*}{H}_n).$$

First we show that E is porous, that is, E is porous at each of its points. Let $x \in E$, then either $x \in C_n$ for some n, or $x \in \overset{*}{H}_n$ for every n, so we have two cases. If $x \in \overset{*}{H}_n$ for all then in particular, $x \in \overset{*}{H}_2$, so $x \notin J_1 \cup C_1$. It follows then that $x \in 2{*}J_1$. As $x \notin \overset{*}{H}_3$, it follows that $x \notin (J_1 \cup J_2) \cup (C_1 \cup C_2)$ and as such $x \in 2{*}J_2$ and so on. Consequently, there is a sequence of intervals (one from J_1, one from J_2, etc.) converging to x with the property that x is in the double of each of these intervals. Hence E is porous at x and the porosity of E at x is at least 1/2. If, on the other hand, there is an n so that $x \in C_n$, then the fact that C_n is the union of symmetric Cantor sets of the form $C(H,n+3)$ and remark (α) made at the conclusion of the second preliminary construction indicates that E is porous at x and the porosity of E at x is at least $1/(n+3)$. Hence E is a porous set.

The sets C_n play an additional important role in our calculation in the following sense. If $x \in C_n$, then

$$x \in \mathrm{int}(\bigcup_{k=1}^{n-1} J_k)^C = \mathrm{int}(\overset{*}{H}_{n-1}) \quad [\mathrm{int} = \mathrm{interior}]$$

and hence these complementary intervals are not used in computing the porosity of E at x. Also, if J_m is any component interval of $\overset{*}{J}_m$ $(m>n)$ then $(n+3){*}J_m \subset O_n$ and $O_n \cap C_n = \emptyset$. Consequently, if one uses only the intervals from $\bigcup_{k=n+1}^{\infty} J_k$ to compute porosity, then the value obtained cannot exceed $1/(n+3)$. However, $C_n \subset (n+3)*J_n$ and consequently the porosity of E at points of C_n must be computed using intervals from J_n. But then remark (β) applies and we conclude that there are points of C_n whose porosity is less than $1/(n+2)$. Thus, E which contains $\bigcup_{n=1}^{\infty} C_n$ contains points of arbitrarily small porosity. The symmetric nature of the construction then allows us to conclude that E has points of arbitrarily small porosity in every portion. Thus, not only is E not globally porous, but no portion of E is globally porous.

Now, suppose E is σ-globally porous. Then $E = \bigcup_{n=1}^{\infty} E_n$ where each E_n is a globally porous set. As E is perfect, one such E_n must be dense in a portion of E so that $\mathrm{cl}(E_n)$ contains that portion. However, PROPOSITION 2. entails that $\mathrm{cl}(E_n)$ is globally porous and this contradicts the fact that no portion of E is globally porous. This completes the proof.

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