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# A Differentiable Function for which Localization for Double Fourier Series Fails

We give an example of a function of two variables, everywhere differentiable and continuously differentiable except at one point, for which square sum localization fails. Igari, [2], has shown that continuous functions of this sort exist. On the other hand, it has been known since the time of Tonelli that, for $n = 2$, localization holds for functions of type BVC, even for rectangular sums. This result has recently been extended to functions of type HBV, [1].

Our example uses only the simplest facts about Fourier series of functions of one variable.

(i) If $D_n$, $n = 1, 2, \ldots$, is the sequence of Dirichlet kernels, and m is a positive integer, there is a constant $c > 0$ such that, for every $n > m$, $\int_{\pi/n}^{\pi/m} |D_n(t)|dt > c \log n/m$.

(ii) If $0 < a < b < \pi$, there is a $c' > 0$ such that, for every n, $\int_a^b |D_n(t)|dt > c' \log \frac{b}{a}$.

We also use the elementary fact that

(iii) cross localization holds for functions of two variables. This means that if there is a $\delta > 0$, and if a summable function f is zero on the cross $([-\pi,\pi] \times [-\delta, \delta]) \cup ([-\delta, \delta] \times [-\pi, \pi])$, then the Fourier series of $f$ converges to zero at the origin (0,0).

We consider continuously differentiable functions $f_i$, $i=1, 2, 3, \ldots$, defined on $I = (-\pi, \pi]^2$ with support in $I_i = [a_i, b_i] \times [\pi/n_i, \pi/i]$ and such that $|f_i(x,y)| < (\pi-b_i)^2$. Let $f = \sum_i f_i$. The strictly increasing sequence of integers $\{n_i\}$, the functions $\{f_i\}$, and the sequences $\{a_i\}$, $\{b_i\}$, with $0 < a_i < b_i < a_{i+1} < \ldots < \pi$ and $b_i \to \pi$, will be defined inductively.

For any $n_k$,

$$\left|\int_I f(x,y)\, D_{n_k}(x)\, D_{n_k}(y)\,dxdy\right| = \left|\sum_i \int_{I_i} f_i(x,y)\, D_{n_k}(x)\, D_{n_k}(y)\,dxdy\right|$$

$$\geq \left|\int_{I_k} f_k(x,y) D_{n_k}(x) D_{n_k}(y)\,dxdy\right| - \left|\sum_{i<k} \int_{I_i} f_i(x,Y) D_{n_k}(x) D_{n_k}(y)\,dxdy\right|$$

$$- \left|\sum_{i>k} \int_{I_i} f_i(x,y) D_{n_k}(x) D_{n_k}(y)\,dxdy\right| = A - B - C .$$

Choose $a_1 = \pi/2$, $b_1 = 3\pi/4$, and $f_1 \equiv 0$. Suppose we have chosen $a_1 < b_1 < \ldots < a_{k-1} < b_{k-1} < \pi$, $2 = n_1 < \ldots < n_{k-1}$, and $f_1, \ldots, f_{k-1}$.

For any choice of $b_k > a_k > b_{k-1}$ and of $n_k > n_{k-1}$, we can choose $f_k$, continuously differentiable, with support in $I_k$, such that $|f_k| < (\pi - b_k)^2$ and $f_k$ approximates

$$(\pi - b_k)^2 \operatorname{sgn}(D_{n_k}(x)\, D_{n_k}(y))$$

so closely that

$$A \geq \frac{1}{2} (\pi - b_k)^2 \int_{\pi/n_k}^{\pi/k} |D_{n_k}(y)|dy \int_{a_k}^{b_k} |D_{n_k}(x)|dx$$

$$\geq c_1 (\pi - b_k)^2 \log(n_k/k) \log b_k/a_k,$$

where $c_1 > 0$ is independent of the choice of $n_k$. Thus, when $a_k$ and $b_k$ have been chosen, we may choose $n_k$ so large that, with $f_k$ as above,

(*) $$A > k + 1.$$

Since $\sum_{i<k} f_i$ vanishes in a cross neighborhood of the origin, for $n_k$ chosen sufficiently large,

(**) $$B < 1/2 .$$

There is a $c_2 > 0$, independent if k, such that

$$C \leq \sum_{i>k} (\pi - b_i)^2 \int_0^{\pi} |D_{n_k}(y)|dy \int_{a_i}^{b_i} |D_{n_k}(x)|dx$$

$$\leq c_2 \sum_{i>k} (\pi - b_i)^2 (\log n_k)(b_i - a_i).$$

If $a_k$ and $b_k$ are chosen with $b_{k-1} < a_k < b_k$, $\pi - 1/k < b_k < \pi$, and so that $(\pi - b_k)^2 (\log n_{k-1})(b_k - a_k) < (c_2 2^k)^{-1}$, then with the inductively determined $\{a_i\}$, $\{b_i\}$, $\{n_i\}$, and $\{f_i\}$,

$$(***) \qquad C \leq \sum_{i>k} (\pi - b_i)^2 (\log n_k)(b_i - a_i)$$

$$< c_2 \sum_{i>k} (\pi - b_i)^2 (\log n_{i-1})(b_i - a_i) < 1/2.$$

Combining the starred estimates we see that

$$\left| \int_I f(x,y)\; D_{n_k}(x)\; D_{n_k}(y)\; dxdy \right| > k$$

for every k, which means that the sequence of square partial sums of the Fourier series of $f$ at $(0,0)$ is unbounded.

It is clear that $f$ is continuously differentiable except at $(\pi,0)$. It is differentiable there since $|f(x,y)| \leq (\pi - x)^2$. This proves the following theorem.

Theorem. There is a function, everywhere differentiable and continuously differentiable except at one point, whose Fourier series does not have square localization.

## References


1. C. Goffman and D. Waterman, The localization principle for double Fourier series, Studia Math. 69 (1980), 41-57.

2. S. Igari, On the localization properly of multiple Fourier series, J. of Approx. Theory, 1 (1968), 182-188.

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