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On Differentiability of Peano Type Functions

In this paper we investigate the properties of Peano functions on the real line $\mathbb{R}$ to the plane $\mathbb{R}^2$, that is, vector functions $F = (f_1, f_2): \mathbb{R} \to \mathbb{R}^2$.

Let $f: A \to \mathbb{R}$ and $E \subset A \subset \mathbb{R}$. We say that f satisfies Banach's condition T_2 on E ($f \in T_2(E)$) if

$$\lambda(\{ y \in f(E): |f^{-1}(\{y\}) \cap E| > \aleph_0 \}) = 0$$

where λ denotes Lebesgue measure. We also shall write $f \in VB(E)$ if f is of bounded variation on E, and $f \in VBG(E)$ if E is the union of a countable sequence of sets E_n on each of which f is of bounded variation. Let M_1 and M_2 be any sets and let $S = M_1 \times M_2$. If $u \in M_1$ and $v \in M_2$, we let $S_u = \{ y \in M_2: (u,y) \in S \}$ and $S^v = \{ x \in M_1: (x,v) \in S \}$. Finally, throughout this note, derivative will mean finite derivative. Our first theorem is:

THEOREM 1. <u>The existence of Peano functions $F=(f_1,f_2)$ such that for each $x \in \mathbb{R}$ at least one of the derivatives f_1' or f_2' exists is equivalent to the Continuum Hypothesis.</u>

The proof uses the following theorem of Sierpinski.

THEOREM S. <u>Let M_1 and M_2 be sets of power $\mathfrak{c}$. The Continuum Hypothesis is equivalent to the existence of a decomposition, $M_1 \times M_2 = S_1 \cup S_2$ where $(S_1)_u$ and $(S_2)^v$ are countable for each $u \in M_1$ and $v \in M_2$.</u>

The following theorems show that even if the mildest conditions are put on the coordinate functions of F, then F cannot be Peano. In particular, it follows from THEOREM 3 that if either f_1 or f_2 is assumed to be measurable, then F cannot be Peano.

THEOREM 2. Let $F=(f_1,f_2)$, where $f_1 \varepsilon T_2(\mathbf{R})$ and f_2 is arbitrary. Then, $\lambda^i(F(\mathbf{R})) = 0$, where λ^i denotes inner Lebesgue measure on $\mathbf{R}^2$.

THEOREM 3. Let f_1 be Lebesgue measurable, and let f_2 be arbitrary. Suppose that for each $x \varepsilon \mathbf{R}$ at least one of the derivatives $f_1'(x)$ or f_2' exists. Then, $\lambda^i(F(\mathbf{R})) = 0$.

To conclude, let us pose the following problem:

QUESTION. Does there exist function $F=(f_1,f_2)$: $I \to I \times I$ where $I=[0,1]$, such that $F(I)=I \times I$ and for each $x \varepsilon I$ at least one of the derivatives f_1' or f_2' exists?

Let us mention that if the I in the above question is taken to be either open or half open, then the existence of such an F is, like in THEOREM 1, equivalent to the Continuum Hypothesis.

REFERENCES

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