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SOME PROPERTIES OF GENERALIZED DERIVATIVES

The intention in this short article is to present a few of the main ideas that have proved useful in the studies [1] and [2]. In order to provide a unified and simplified way of looking at a number of generalized derivatives one can consider the following simple scheme: for each $x\varepsilon R$ suppose that one is given a collection of sets $S(x)$ having the properties (i) if $\sigma\varepsilon S(x)$ then $x\varepsilon\sigma$, (ii) $\{x\}\notin S(x)$, (iii) if $\sigma\varepsilon S(x)$ and $\sigma'\supset\sigma$ then $\sigma'\varepsilon S(x)$, and (iv) if $\sigma\varepsilon S(x)$ and $\eta>0$ then $\sigma\cap(x-\eta,x+\eta)\varepsilon S(x)$. For such a system S one defines derivates and derivatives of a function F in the familiar way. We write

$$\underline{D}_S F(x) = \sup_{\sigma\varepsilon S(x)} \inf \{[F(y)-F(x)]/[y-x] : y\varepsilon\sigma,\ y\neq x\}$$

and similarly for the upper derivate while for derivatives one writes $D_S F(x) = f(x)$ if the set

$$\{y : |F(y)-F(x)-f(x)(y-x)| \leq \eta|y-x|\}$$

belongs to $S(x)$ for all positive numbers η.

Depending, obviously, on properties possessed by the system S the derivates and derivatives with respect to S will have certain properties and the goal in such a study is to determine natural assumptions to make that yield an attractive theory. Such a scheme has been considered in the past by numerous authors, usually with more structure than required here, and so it is not so obvious what type of assumptions should be taken. In the articles [1] and [2] we have found that almost all of the familiar properties of well-known derivatives follow basically from only two elementary types of assumptions: porosity assumptions and assumptions about the way in which the sets in the collections $S(x)$ must intersect. Here we present just two theorems that illustrate our main themes, but many more are developed in the papers cited.

Our first theorem illustrates how a porosity assumption can be used in this type of approach; a proof is given in [2]. We use the notation $D^\# F(x)$ to denote the sharp (or strong) derivates of the function F defined as

$$\underline{D}^\# F(x) = \lim\inf \{[F(y)-F(z)]/(y-z) : (y,z)\to(x,x), y\neq z\}.$$

Then our theorem is a generalization of a well-known theorem of

W.H.Young [3] for the Dini derivatives of a continuous function.

THEOREM *Let F be a continuous function and suppose that for each* $x \in R$ *and each* $\sigma \in S(x)$ *the set* σ *has either right or left porosity less than 1. Then for every* x *excepting possibly a set of the first category one has* $\underline{D}_S F(x) = \underline{D}^{\#} F(x)$.

A number of other theorems depend not on the thickness of the sets in $S(x)$ but more on the way in which pairs of sets intersect. The system S is said to satisfy an intersection condition provided for any choice of system $\{\sigma_x : x \in R\}$ with each $\sigma_x \in S(x)$ there is a positive function δ so that whenever $0 < y-x < \min\{\delta(x), \delta(y)\}$ the sets σ_x and σ_x intersect in some prescribed fashion. As an illustrative example we will use the intersection requirement that

$$\sigma_x \cap \sigma_y \cap (-\infty, x] \quad \text{and} \quad \sigma_x \cap \sigma_y \cap [y, +\infty)$$

are both nonempty. With this intersection condition one can prove the following theorem; a more general statement is proved in Bruckner, O'Malley and Thomson [1].

THEOREM *Suppose that the system* $\{S(x) : x \in R\}$ *satisifies the above intersection condition and that the function* F *has everywhere an S-derivative* f. *Then* f *is in the first class of Baire*.

The articles [1] and [2] use a simplified version of this type of derivative but the proofs can be carried to this setting with only some notational changes. There are many more results that will appear in these articles but these two theorems should be enough to indicate the intention and the flavour of the theory we have so far developed.

REFERENCES

[1] A.M.Bruckner, R.J.O'Malley, and B.S.Thomson, Path derivatives: a unified view of certain generalized derivatives, (submitted).

[2] B.S.Thomson, Some theorems for extreme derivates, Journal London Math. Soc. (to appear).

[3] W.H.Young, Oscillating sequences of continuous functions, Proc. London Math. Soc. 6 (1908), 298-320.