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Riemann-Stieltjes Integrals of Functions of Generalized Bounded Variation

This work appears in [3].

We define a new notion of generalized bounded variation as follows: We shall say that a sequence of convex functions $\Phi = \{\phi_n\}$ is a <u>Φ-sequence</u> if (i) $\phi_n(0) = 0$ and $\phi_n(x) > 0$ for $x > 0$, $n=1,2,\ldots$, (ii) $\phi_{n+1}(x) \leq \phi_n(x)$ for $x \geq 0$, $n=1,2,\ldots$, and (iii) $\sum_n \phi_n(x) = \infty$ for $x > 0$. A real-valued function f is said to be of <u>Φ-bounded variation</u> on the interval $[a,b]$ if $\sum_n \phi_n(|f(I_n)|) < \infty$ for any collection $\{I_n\}$ of nonoverlapping subintervals of $[a,b]$, where $f([x,y]) = f(y) - f(x)$. The space <u>ΦBV</u> is the collection of all functions f such that cf is of Φ-bounded variation for some $c > 0$. Finally <u>ΦBV_0</u> $= \{f \in \Phi BV : f(a)=0\}$. This definition of Φ-bounded variation is equivalent to requiring that the sums be uniformly bounded or that the sums obtained from finite collections of subintervals be uniformly bounded. This being the case, we may define the <u>total Φ-variation</u> of f over $[a,b]$ by $V_\Phi(f;a,b) = V_\Phi(f) = \sup \sum_n \phi_n(|f(I_n)|)$, the supremem being taken over all nonoverlapping collections $\{I_n\}$. In turn we define the <u>Φ-variation function</u> of f, for $a \leq x \leq b$, $v_\Phi(x;f) = v_\Phi(x) = V_\Phi(f;a,x)$.

By varying this construction, we obtain many of the spaces of functions of generalized bounded previously studied. For example, by choosing $\phi_n(x) = x$ for all n, we obtain ordinary bounded variation. If $\phi(x)$ is a convex function of the type described above, and if we let $\phi_n(x) = \phi(x)$, $n=1,2,\ldots$, we have $\Phi BV = \phi BV$, as studied by Young [6]. If $\Lambda = \{\lambda_n\}$ is a Λ-sequence in the sense of Waterman [4], and we let $\phi_n(x) = x/\lambda_n$, we have $\Phi BV = \Lambda BV$, and if ϕ and Λ are as above and $\phi_n(x) = \phi(x)/\lambda_n$, we have $\Phi BV = \phi\Lambda BV$ (see [2]).

We have generalized many of the known results regarding the behavior of functions in these classes and the associated variation functions. For example, for $a < c < b$, we have $V_\Phi(f;a,b) \leq V_\Phi(f;a,c) + V_\Phi(f;c,b) + \phi_1(\mathrm{osc}(f))$. Since V_Φ has all of the variations alluded to above as special cases, it is clear that no more can be said without specific knowledge of the sequence Φ. However the variation function v_Φ shares the continuity properties of f, as was known for ordinary bounded variation and Λ-bounded variation, that is, v_Φ is (right- or left-) continuous at $x \varepsilon [a,b]$ if and only if f is (right- or left-) continuous there.

It is not difficult to see that ΦBV is a linear space, and that a function of Φ-bounded variation can have only simple discontinuities. Further, ΦBV_0 can be made a Banach space with an appropriate norm, and the following version of the Helly Selection Theorem holds:

> If $\{f_n\}_{n=1}^{\infty} \subseteq \Phi BV$ such that there exist $c > 0$ and $M < \infty$ with $|cf_n(x)| < M$ for all n and $x \varepsilon [a,b]$ and $V_\Phi(cf_n) < M$ for all n, then there is a subsequence $\{f_{n_k}\} \subseteq \{f_n\}$ and a function $f \varepsilon \Phi BV$ so that $f_{n_k}(x) \to f(x)$ for all $x \varepsilon [a,b]$ and $V_\Phi(cf) \leq M$.

We also obtain the following analogue of a theorem of Waterman: Recall [5] that if f is of harmonic bounded variation, then the Fourier series of f converges for all x, and the convergence is uniform on closed intervals of points of continuity, and that if ΛBV is strictly larger than HBV, there is a continuous function in ΛBV whose Fourier series diverges at a point. We have shown that this last statement remains true if "ΛBV" is replaced with "ΦBV".

Our main theorem is the following generalization of a theorem of Leśniewicz and Orlicz [1]:

Let $\Phi = \{\phi_n\}$ and $\Psi = \{\psi_n\}$ be Φ-sequences such that $\sum_k \phi_k^{-1}(1/k)\psi_k^{-1}(1/k) < \infty$. If $f \varepsilon \Phi BV_0 \cap C$ and $g \varepsilon \Psi BV$, then the Riemann-Stieltjes integral $\int_a^b f\,dg$ exists.

As a corollary to the proof we obtain the following Hölder-like inequality:

Let $\Lambda = \{\lambda_k\}$ and $\Gamma = \{\gamma_k\}$ be Λ-sequences such that $\sum_k \lambda_k \gamma_k / k^2 < \infty$. For $f \varepsilon \Lambda BV_0 \cap C$ and $g \varepsilon \Gamma BV_0$, the following holds:

$$\left|\int_a^b f\,dg\right| \leq 2\|f\|_\Lambda \|g\|_\Gamma \sum_k \lambda_k \gamma_k / k^2 .$$

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