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# Strong Porosity Features of Typical Continuous Functions

A number of theorems in the literature deal with the differentiability structure of the typical continuous function. Most of these theorems are negative - the typical continuous $f$ is nowhere differentiable, nowhere approximately differentiable, nowhere symmetrically differentiable and nowhere approximately symmetrically differentiable. Some of the theorems are positive - see the theorems of Marcinkiewicz, Jarnik and Scholz discussed in [1] and [2]. Certain other theorems deal with the level set structure of the typical $f$ [3][6].

In [4] we prove a general theorem with a number of consequences. Certain of the negative results follow readily from these consequences. In addition, these consequences provide additional insights for the positive theorems and for the theorems concerning level sets. Finally, certain other results which hold for all continuous functions become more precise in light of our results.

A set $E$ is called bilaterally strongly porous, bsp, at a point $x$ if the left and right porosities of $E$ at $x$ both equal 1. If $E$ is bsp at all its points, we say that $E$ is bilaterally strongly porous.

Main Theorem, Let $\mathcal{K}$ be a $\sigma$-compact subset of $C = C[0,1]$(furnished with the sup norm). Let $\mathfrak{F}$ consist of those functions $f \in C$ such that $\{x: f(x) = g(x)\}$ is bsp for all $g \in \mathcal{K}$. Then $\mathfrak{F}$ is residual in $C$.

We discuss some consequences.

(1) Let $\mathcal{L}_\alpha$ consist of those functions which satisfy a Lipschitz condition of order $\alpha$ $(0 < \alpha \leq 1)$. It is easy to verify that $\mathcal{L}_\alpha$ is $\sigma$-compact, so the theorem applies.

Consideration of the case $\alpha = 1$ leads to the fact that the typical f can agree with functions possessing bounded derivatives only on sets which are bsp.

For $\alpha < \frac{1}{2}$ B. ϕksendal has observed that we obtain an interesting comparison of typicalness in our sense and typicalness in the sense of Brownian motion. Our typical f will agree with functions in $\mathcal{L}_\alpha$ on only bsp sets. On the other hand, typical functions in the sense of Brownian motion are in $\mathcal{L}_\alpha$ for every $\alpha < \frac{1}{2}$.

(2) The case $\alpha = 1$ gives information only about functions possessing bounded derivatives. The class of functions possessing finite derivatives is not σ-compact. Nonetheless, a simple argument shows that a typical f can agree with a differentiable function only on a set which is bsp. It follows readily that the typical f must be nowhere differentiable, nowhere approximately differentiable and nowhere preponderantly differentiable (even unilaterally). Moreover the limit $\lim_{n\to\infty} \frac{f(x+h_n)-f(x)}{h_n}$ cannot exist (finite) at any point unless the sequence $\{h_n\}$ approaches 0 so rapidly as to be bsp at 0.

(3) The intersection pattern of the graph of f with the family of nonvertical lines was studied in detail in [3]. In particular, most such intersections (if nonempty) contain perfect sets. The theorem shows all these perfect sets are bsp. For horizontal lines, this had already been observed by Thomson [6].

(4) Recently, Laczkovich [5] has shown that if P is a perfect set of positive measure and f is continuous there is a perfect subset $Q \subset P$ for which $f|Q$ is infinitely differentiable in one of two ways. Only one of these two ways is possible for the typical f, namely, that there exists a positive integer n such that $(f|Q)^{(n)} \equiv 0$. The set Q must be bsp, since $(f|Q)^{(n-1)}$ must be differentiable.

Finally, we show that the typical f exhibits some regularity behavior that certain well-behaved functions don't exhibit - the upper and lower convex boundaries of the typical f possess finite derivatives in (0,1) and infinite derivatives at the end points. (These derivatives are unbounded Cantor-like functions).

## REFERENCES


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