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A Generalization of Density Topology.

The similarities and differences between measure and category have been studied by many mathematicians. A good deal of information about problems of this kind is collected in the excellent book by Oxtoby [3]. According to my best knowledge there is not in the literature a good concept of the category analogue of a density point of a set. The notion of a qualitative point, which has been used from time to time when studying derivatives ([2]) or cluster sets ([6]), does not appear to be very delicate (see for example [1], p. 166).

This note attempts to formulate a concept of a $\mathcal{J}$-density point of a set for an arbitrary σ-ideal $\mathcal{J}$, which will reduce for the σ-ideal of null sets to the notion of a density point and which will give for the σ-ideal of meager sets a quite satisfactory and delicate new notion, which can be considered as a starting point for studying "category" approximate continuity, differentiability and so on. Here we shall present only some basic definitions and properties. More detailed exposition will be included in [7] (general σ-ideals) and [4] (the σ-ideal of meager sets).

Let $(X, \mathcal{S}, m)$ be a finite (or σ-finite) measure space. A sequence $\{f_n\}_{n \in N}$ of $\mathcal{S}$-measurable real functions defined on X converges to a function f if and only if every subsequence $\{f_{m_n}\}_{n \in N}$ of $\{f_n\}_{n \in N}$ contains a subsequence $\{f_{m_{p_n}}\}_{n \in N}$ converging to f almost everywhere. This well known fact allows us to introduce a generalization of the notion of convergence in measure of sequences of measurable functions (see [5]): Let $(X, \mathcal{S})$ be a measurable space. Let $\mathcal{J} \subset \mathcal{S}$ be a proper σ-ideal. We shall say that some property holds $\mathcal{J}$-almost everywhere ($\mathcal{J}$-a.e.) if and only if the set of points which do not have this property belongs to $\mathcal{J}$. We shall say that the sequence $\{f_n\}_{n \in N}$ of $\mathcal{S}$-measurable real functions defined on X converges with respect to $\mathcal{J}$ to the $\mathcal{S}$-measurable real function f defined on X if and only if every subsequence $\{f_{m_n}\}_{n \in N}$ of $\{f_n\}_{n \in N}$ contains a subsequence $\{f_{m_{p_n}}\}_{n \in N}$ converging to f $\mathcal{J}$-a.e.. We shall use the denotation $f_n \xrightarrow[n\to\infty]{\mathcal{J}} f$.

Observe that the definition of a density point of a set can be formulated using only convergence in measure in the following way: 0 is a point of density of A if and only if the sequence $\{\chi_{(n \cdot A) \cap [-1,1]}\}_{n \in N}$ of characteristic functions (where $n \cdot A = \{nx: x \in A\}$) converges in measure to 1 on the interval $[-1,1]$. Now it is clear how to define a notion of $\mathcal{J}$-density point for an arbitrary σ-ideal $\mathcal{J}$:

Def. 1. We shall say that 0 is a $\mathcal{J}$-density point of a set $A \subset R$ if and only if $\chi_{(n \cdot A) \cap [-1,1]} \xrightarrow[n\to\infty]{\mathcal{J}} 1$.

Similarly one can define right- or left-hand $\mathcal{J}$-density at 0, and 0 is a $\mathcal{J}$-dispersion point of A if and only if the limit is 0.

Obviously we can take some interval $[-a,a]$, $a > 0$, instead of $[-1,1]$. Other modifications are also possible (see [7] or [4]).

Def. 2. We shall say that x_o is a $\mathcal{J}$-density point of A if and only if 0 is a $\mathcal{J}$-density point of $A - x_o = \{x-x_o : x \in A\}$.

In the sequel we can consider only sets having the Baire property as the σ-algebra $\mathcal{S}$ and $\mathcal{J}$ will always denote the family of meager sets on the real line R. If $A \Delta B \in \mathcal{J}$ (Δ means the symmetrical difference), then we shall write $A \sim B$. Denote $\Phi(A) = \{x \in R: x \text{ is a } \mathcal{J}\text{-density point of } A\}$.

Th. 1. for every $A, B \in \mathcal{S}$

1) $\Phi(A) \sim A$,

2) if $A \sim B$, then $\Phi(A) = \Phi(B)$,

3) $\Phi(\emptyset) = \emptyset$, $\Phi(R) = R$,

4) $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

Remark. From 1) it follows that if $A \in \mathcal{S}$, then $\Phi(A) \in \mathcal{S}$.

Th. 2. The family $\mathfrak{T}_{\mathcal{J}} = \{\Phi(A) - N: A \in \mathcal{S}, N \in \mathcal{J}\}$ is a topology on the real line.

The topology $\mathfrak{T}_{\mathcal{J}}$ will be called the $\mathcal{J}$-density topology (or qualitative density topology).

Remark. It is not difficult to observe that $\mathfrak{T}_{\mathcal{J}} = \{A \in \mathcal{S}: A \subset \Phi(A)\}$.

Th. 3. There exists an open set $E = \bigcup_{n=1}^{\infty} (a_n, b_n)$ such that $b_n \searrow 0$, the intervals (a_n, b_n) are pairwise disjoint and 0 is a $\mathcal{J}$-dispersion point of E.

From the above theorem it follows that the set having x_o as a $\mathcal{J}$-density point need not be residual at any neighborhood of x_o. This fact we had had in mind writing earlier that this notion is a delicate one.

Th. 4. $\mathfrak{T}_{\mathcal{J}}$ is T_2 but not T_3.

We conclude this note with some information on $\mathcal{J}$-approximately continuous functions.

Def. 3. A function $f:R\to R$ is called $\mathcal{J}$-approximately continuous at x_o if and only if for every $\varepsilon > 0$ the set $f^{-1}((f(x_o) - \varepsilon, f(x_o) + \varepsilon))$ has x_o as a point of $\mathcal{J}$-density.

Def. 4. A function $f:R\to R$ is called $\mathcal{J}$-approximately continuous if and only if for every interval (y_1, y_2) the set $f^{-1}((y_1, y_2))$ belongs to $\mathfrak{T}_{\mathcal{J}}$.

In other words, a $\mathcal{J}$-approximately continuous function is a continuous function from $(R, \mathfrak{T}_{\mathcal{J}})$ into R equipped with the natural topology (and is $\mathcal{J}$-approximately continuous at every point).

Th. 5. A function f has the Baire property if and only if it is $\mathcal{J}$-approximately continuous $\mathcal{J}$-a.e..

Th. 6. If $f:R\to R$ is $\mathcal{J}$-approximately continuous, then it is of Baire class 1 and has the Darboux property.

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