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Finitistic real analysis

We let ω_p (p any rational number) be symbols which will act like positive integers but will satisfy an infinite set of axioms

$$(*) \qquad 2^{\omega_p} < \omega_q$$

where p , and q are arbitrary rational constants with $p < q$. We can think and work with the ω_p's as if they were very large integers, since in every proof only finitely many of the axioms (*) may be used. We put

$$\mathbb{R}_p = \{ \frac{k}{\omega_p!} : |k| \leq (\omega_p!)^2 \} \quad ,$$

$$\zeta_p = \frac{1}{\omega_p} \quad ,$$

$$x =_p y \Leftrightarrow |x - y| < \zeta_p \quad .$$

For additional motivation of those axioms and concepts and some developments see [1,2] . E.g. our definition of the $\mathbb{R}_p$'s secures $\mathbb{R}_p \subseteq \mathbb{R}_q$ for $p \leq q$.

Definitions. $s < r < p$, $s < q$, $f : \mathbb{R}_p \to \mathbb{R}_q$.

(i) f is said to be rs - continuous iff

$$x =_r y \Rightarrow f(x) =_s f(y) \quad ,$$

for all $x,y \in \mathbb{R}_p$.

(ii) f is said to be rs - measurable iff

$$\Pr(f(x) =_s f(y) \mid x =_r y) =_r 1 \quad ,$$

where the conditional probability is computed by counting.

(iii) A set $X \subseteq \mathbb{R}_p$ is said to be rs-measurable iff its characteristic function is rs-measurable.

(iv) For every $X \subseteq \mathbb{R}_p$ we put

$$\mu(X) = \frac{|X|}{\omega_p!}$$

(v) We put

$$\int_a^b f = \int_a^b f(x)dx = \begin{cases} \frac{1}{\omega_p!} \sum_{a \le x < b} f(x) & \text{for } a \le b \quad , \\ -\frac{1}{\omega_p!} \sum_{b \le x < a} f(x) & \text{for } a > b \quad . \end{cases}$$

(vi) f is said to be <u>rs-differentiable</u> at x iff

$$\frac{f(x) - f(x')}{x - x'} =_s \frac{f(x) - f(x'')}{x - x''}$$

whenever $x' =_r x =_r x''$, $x' \neq x \neq x''$.

(vii) We put for $x \in \mathbb{R}_p$, $x < \omega_p!$

$$f'(x) = \frac{d}{dx} f(x) = \omega_p!\,(f(x + \frac{1}{\omega_p!}) - f(x))$$

With these definitions we can begin to develop a finitistic variant of real analysis. This project has not been accomplished yet. Our (v) and (vii) imply the identities

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \quad \text{and} \quad \int_a^b f'(t)dt = f(b) - f(a) \quad .$$

But the Lebesgue theory of integrability and differentiability which is a kind of stability theory (see (ii) and (vi)) remains to be developed. A new kind of differentiability (different from (vi)) may be required to establish a counterpart of Lebesgue's theorem about almost everywhere differentiability of monotonic functions.

We can define

$$e^x = \sum_{k=0}^{\omega_0} \frac{x^k}{k!}$$

and prove its usual properties and numerical estimates, without losing the comfort of working with a finite sum. Another advantage is that the space of functions $f : \mathbb{R}_p \to \mathbb{R}_q$ is finite which permits to state and to prove (by simple counting) the following theorem (see [2]).

<u>Theorem</u>. $(s < r < p\ ,\ s < q\ ,\ u < t < p\ ,\ u < q)$ The probability that an rs - continuous function $f : \mathbb{R}_p \to \mathbb{R}_q$ is tu - differentiable at some $x \in \mathbb{R}_p$ is less than ζ_p .

Until now the great building of analysis was founded on infinite objects (even when constructivists were the builders [0]). We believe that this has happened by accident and not by rational choice. This accident gave analysis admirable qualities which lead to the building of set theory. Such developments would not have occurred if analysis had started in the way which we propose here. Still it may be interesting, and, as the above Theorem shows, perhaps not quite routine to explore new questions of analysis which we can formulate using our potential infinities ω_p and potential zeros ζ_p . (Some applications to logic are given in [3] .)

References

[0] E. Bishop, Foundations of constructive analysis, New York 1967.

[1] J. Mycielski, Analysis without actual infinity, Notices of the A.M.S. 26 (1979), A-523.

[2] J. Mycielski, Analysis without actual infinity, Journal of Symbolic Logic, to appear.

[3] J. Mycielski, Consistency proofs in FIN , Abstracts of the A.M.S., to appear.

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