Real Analysis Exchange Vol. 6 (1980-81)

M. Laczkovich and G. Petruska, Department I of
Analysis, ELTE, Budapest, Hungary

REMARKS ON A PROBLEM OF A. M. BRUCKNER

1. Let F be a continuous function on $(0,1)$ and suppose that, for a given sequence $h_n \to 0$, $h_n \neq 0$ the finite limit

$$(1) \qquad \lim_{n\to\infty} \frac{F(x+h_n) - F(x)}{h_n} = f(x)$$

exists for every $0 < x < 1$. In this case F is said to be sq -differentiable and f is its sq -derivative /"sq" stands for "sequential"/. An sq -derivative is obviously a Baire 1 function and the problem we refer to in the title is whether every Baire 1 function is the sq -derivative of a suitable F /see [1], pp. 115-117./. We answer this question in the negative. It turns out, that the property "being sq -differentiable" is a rather restrictive one. For instance, if the sq -derivative f of F is bounded then F is absolutely continuous /Lipschitz 1/, and hence $f(x) = F'(x)$ a.e.

Notation. For any function F, D^+F, D_+F, D^-F, D_-F

denote the usual derived numbers. The Lebesgue measure is denoted by λ . The set A translated by x is denoted by $A+x$.

2. The results below are easy consequences of a known theorem on Perron integral /[2], Theorem (7,3), pp. 204-205./.

<u>Theorem 1.</u> Let f be a sq-derivative of the continuous function F .

(i) If $f \geq 0$, then F is increasing;

(ii) if f is summable, then

$$F(x) = \int^{x} f + \text{const},$$

and hence $F' = f$ a.e. In particular, if f is bounded, then F is Lipschitz 1;

(iii) if $f(x) = g(x)$ a.e., where g is an ordinary derivative, then $f(x) = g(x)$ everywhere and $F'(x) = f(x)$, i.e. F is a primitive of f ;

(iv) there always exists an everywhere dense open set U such that $F'(x) = f(x)$ a.e. in U .

<u>Theorem 2.</u> Let f and g be the sq-derivatives /with respect to the same sequence $\{h_n\}$/ of the continuous functions F and G respectively. If $f(x) = g(x)$ a.e., then $F-G$ is constant.

Theorem 3. Let f be a Baire 1 function such that either

(i) f is summable but $(\int^x f(t)dt)'_{x=x_0} \neq f(x_0)$ in a suitable point x_0, or

(ii) there exists a derivative g such that the non-empty set $\{x: f(x) \neq g(x)\}$ is of measure zero.

Then f can not be a sq-derivative. In particular, changing the values of a derivative in finite number of points, the result is a Baire 1 function which is not a sq-derivative.

Example 4. Let f be a bounded function and suppose it is right continuous in every point x. Then taking $F(x) = \int^x f$, we have

$$D^+F(x) = D_+F(x) = f(x)$$

in every x. In particular f is the sq-derivative of F for any positive null-sequence $\{h_n\}$. Taking an increasing function f with jumps, the example above is a non-Darboux sq-derivative.

Theorem 5. If the sequence $h_n \to 0$ contains infinitely many positive as well as negative terms then the sq-derivatives with respect to $\{h_n\}$ possess Darboux property.

3. The next assertions elucidate how sharp are the results above.

<u>Theorem 6.</u> Let a nowhere dense perfect set $P \in [0,1]$, a sequence $h_n \to 0$, $h_n \neq 0$ be given. Then there exists a continuous function F such that its sq-derivative /with respect to $\{h_n\}$/ exists everywhere, but the ordinary derivative F' fails to exist in the points of P.

<u>Theorem 7.</u> Let f be the sq-derivative of F with respect to $h_n \to 0$ and suppose $h_n > 0$, $\frac{h_n}{h_{n+1}} \to 1$. Then $F'_+(x) = f(x)$ holds on an everywhere dense open set /F'_+ denotes the right hand side derivative of F/.

<u>Corollary 8.</u> If the sequence $h_n \to 0$ contains two subsequences $h_n^{(1)}, h_n^{(2)}$ such that $h_n^{(1)} > 0$, $h_n^{(2)} < 0$ and

$$\lim \frac{h_n^{(1)}}{h_{n+1}^{(1)}} = \lim \frac{h_n^{(2)}}{h_{n+1}^{(2)}} = 1 ,$$

then $F'(x) = f(x)$ holds on an everywhere dense open set.

Our next theorem points out that theorem 7 holds no longer true without the restriction $h_n/h_{n+1} \to 1$.

<u>Theorem 9.</u> Let $h_1 > h_2 > \dots$ be a sequence tending to zero, but $\frac{h_n}{h_{n+1}} \to 1$. Let $H = \{r_1, r_2, \dots\}$ be any

countable set in [0,1]. Then there exists a continuous function F such that its sq-derivative exists everywhere but the ordinary derivative F' fails to exist in the points of H.

4. The definition of the sq-derivative resembles in many respect to that of the selective derivative introduced by O'Malley /[3], or [1], p. 170/. There are many properties possessed by both derivatives. They are however very much unlike in the following sense. If F has selective derivative with respect to two selections then the derivatives agree except on a countable set; in addition to this a selective derivative of F is always the approximate derivative of F in almost every point. Our next theorem shows that neither of these properties hold for sq-derivatives.

Theorem 10. Let $\varepsilon > 0$ be given. Then there exist a perfect set $P \subset [0,1]$, a function F continuous on $[0,1]$, and two sequences $h_n \to 0$, $k_n \to 0$ such that

(i) $\lambda(P) > 1-\varepsilon$;

(ii) the finite limits

$$f(x) = \lim_{n\to\infty} \frac{F(x+h_n) - F(x)}{h_n},$$

$$g(x) = \lim_{n\to\infty} \frac{F(x+k_n) - F(x)}{k_n}$$

exist everywhere;

(iii) $f(x) \neq g(x)$ for $x \in P$,

(iv) F'_{app} does not exist in the density points of P.

5. Problems.

(i) Let the continuous functions F and G be sq - differentiable on the sequences $\{h_n\}$ and $\{k_n\}$, respectively. Suppose that their sq -derivatives agree everywhere. Does $F-G$ constant follow? Corollary 2 and 3 imply that $F-G$ is locally constant on an everywhere dense open set.

(ii) What assumptions on $\{h_n\}$ could imply that sq -differentiable functions have approximate derivatives almost everywhere?

(iii) Suppose that F is sq -differentiable on $\{h_n\}$ as well as on $\{k_n\}$ and $h_n/k_n \to 1$. Do the sq - derivatives agree almost everywhere?

(iv) Is the class of all sq -derivatives additive? uniformly closed? a Borel set in the space of Baire 1 functions /with the topology of uniform convergence/? It is routine from Corollary 2 (ii) that the set of sq - derivatives with respect to a fixed sequence is uniformly closed.

References

[1] A.M. Bruckner, Differentitation of Real Functions, Springer Verlag, 1978 /Lecture Notes in Math. 659/

[2] S. Saks, Theory of the integral, Dover Publications, Inc., New York

[3] R. O'Malley, Selective Derivatives, Acta Math. Acad. Sci. Hung., 29 /1977/, 77-97.

Received June 20, 1980