Real Analysis Exchange Vol. 6 (1980-81)

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SOME SET THEORETIC PROPERTIES OF σ-POROUS SETS

Several theorems in analysis utilize the dual notions of measure zero and first category to describe sets which are exceptional to some particular assertion. Indeed, should an exceptional set prove to be both of measure zero and of the first category, it is generally thought of as being small. Recently, and for the purpose of tightening the theory, E.P. Dolzenko [1] introduced the more restrictive notion of σ-porosity. Specifically, if S is a set, (for purpose of exposition, we assume our domain is the real line, R) the porosity of S at a point x is defined as

$$p(S,x) = \limsup_{\delta\to 0} \ell(S,x,\delta)/\delta$$

where $\ell(S,x,\delta)$ denotes the length of the longest interval in $(R-S)\cap(x-\delta,\ x+\delta)$. The set S is porous at x if $p(S,x)>0$, S itself is porous if it is porous at each of its points, and S is σ-porous if it is the countable union of porous sets. Although it is evident that each σ-porous set is of measure zero and the first category, the reverse implication is not true, and this was demonstrated by L. Zajicek [2], where a general account of various generalizations of porosity are given. Several of the classical theorems involving exceptional sets have been improved using this more restrictive notion.

The purpose of this note is to examine some of the set theoretic properties of σ-porous sets, and to compare these with analogous properties of measure zero, first category sets.

The following three facts concerning measure zero, first category sets are well known:

A. *Every measure zero, first category set is contained in* $G_{\delta\sigma}$ *measure zero, first category set.*

B. *Some measure zero, first category sets are contained in no* G_δ *first category set.*

C. *Some measure zero, first category sets are contained in no* F_σ *measure zero set.*

Although A, B, and C accurately describe the Borel structure of the *combined* notion of measure zero first category, in practice it is often the separate features of the individual notions which prove most useful. That is, an exceptional set is shown to be of the first category by using the fact that every first category set is contained in an F_σ first category set. Subsequently, this same exceptional set is shown to be of measure zero by using an enveloping G_δ set of measure zero. It is unfortunate, but perhaps not surprising that the Borel structure of σ-porous sets closely parallels that of the combined notion of measure zero first category individual Borel tools are not available. In particular we prove that:

A'. Every σ-porous set is contained in a $G_{\delta\sigma}$ σ-porous set.

B'. Some σ-porous sets are contained in no G_δ first category set.

C'. Some σ-porous sets are contained in no F_σ measure zero set.

Proof of A'.

It is evident that we need only show that every porous set is contained in a $G_{\delta\sigma}$ σ-porous set. Let E be a porous set and for each natural number k define $f_k : R \to R$ by $f_k(x) = \ell(E, x, 1/k)$. The function f_k is evidently continuous. Next let

$$g_k(x) = \sup_{i \geq k} \{i \cdot f_i(x)/2\}$$

and note that g_k is a Baire class one function. Also, as the function sequence $\{g_k\}$ is bounded and decreasing, it has a limit, say g, and g is a Baire class two function. But for every $x \in R$, $g(x) = p(E,x)$ (the porosity of E at x). Hence, $E \subseteq g^{-1}(0,\infty)$ and this set is a $G_{\delta\sigma}$. Furthermore, the set $F = g^{-1}(0,\infty) \cap cl(E)$ [cl≡closure] is $G_{\delta\sigma}$ and is porous.

Proof of B'.

The rational numbers suffice.

Proof of C'.

We will actually construct a porous set which is contained in no F_σ measure zero set. Let E be a nowhere dense perfect set of positive measure whose set of points of density, E_0, is dense in E and let $E^* = \{x \in E : p(E,x) \geq \frac{1}{2}\}$. Then E^* is dense in E.

For each $x \in E^*$ and each natural number n, let δ_x^n be a number such that

$$0 < \delta_x^n < 2^{-n} \text{, and}$$

$$\ell(E,x,\delta_x^n) > 2\delta_x^n/3 .$$

Then there is a number $\varepsilon_x^n > 0$ such that if $y \in (x-\varepsilon_x^n, x+\varepsilon_x^n)$ we have that $\ell(E,y,\varepsilon_x^n) > 2\varepsilon_x^n/3$. Let

$$G_n = \bigcup_{x \in E} (x-\varepsilon_x^n, x+\varepsilon_x^n)$$

and define $D = \bigcap_{n=1}^{\infty} G_n \cap E$. Then D is a G_δ set and D has porosity greater than 1/3 at each of its points. In addition, D is a residual set in the relative topology of E .

Now suppose that $D \subseteq \bigcup_{n=1}^{\infty} F_n$ when each F_n is closed and measure zero. Then there is an index n_0 and open interval I such that $I \cap E \neq \emptyset$ and $I \cap F_{n_0}$ is dense in $I \cap E$. But, F_{n_0} is closed, and so $I \cap F_{n_0} = I \cap E$. However, this contradicts the fact that E_0 is dense in E .

Note. While the rational numbers suffice to show that some σ-porous sets are contained in no G_δ σ-porous set, we have not been able to answer the following:

Problem. Does there exist a porous set which is contained in no σ-porous G_δ set?

References

1. E.P. Dolzenko, "Boundary Properties of Arbitrary Functions", Izv. Akad Nauk SSSR, Ser. 31 (1967) No. 1, p. 1-12.

2. L. Zajicek, "Sets of σ-Porosity and sets of σ-porosity (q)", Cas. pro pest. mat. 101 (1976), p. 350-359.

Received June 13, 1980