Real Analysis Exchange Vol. 6 (1980-81)
B.S. Thomson, Department of Mathematics, Simon Fraser University, Burnaby, B.C., Canada V5A 1S6

ON FULL COVERING PROPERTIES

The notion of a Vitali cover plays a fundamental role in the study of derivatives; a simpler but similar role may be played by the notion of a full cover. A collection $\mathcal{C}$ of closed intervals is said to be a Vitali cover of a set X if at each point x in X there are in $\mathcal{C}$ intervals containing x of arbitrarily small length; such a collection is said to be a full cover if moreover at each point x there is a positive number $\delta(x)$ such that $\mathcal{C}$ includes every interval containing x and with length smaller than $\delta(x)$. Both of these concepts are local in nature and both possess properties that are global. The celebrated theorem of Vitali provides an important global property for the former concept. For the full covers a considerably simpler property is available: if $\mathcal{C}$ is a full cover of an interval [a,b] then $\mathcal{C}$ contains a partition of every subinterval of [a,b].

To illustrate how such a feature may be used consider that we are given a function f on [a,b] whose lower extreme derivate is everywhere positive. Then it is easy to check that the collection of all intervals I for which $f(I) > 0$ is a full cover of [a,b];

thus given any subinterval J of $[a,b]$ we may extract from this cover a partition $\{I_i\}$ of J and obtain $f(J) = \Sigma f(I_i) > 0$ from which we can conclude that f is strictly increasing on $[a,b]$. This then gives a particularly revealing proof of a well-known fact.

This idea of extracting partitions from full covers has not been much exploited, although it can be traced back at least to the beginnings of this century (*cf.* Goursat [6]) and it has played a central role in certain theories of integration (*eg.* Henstock [7]). In this note we discuss several types of full covers, the partitioning properties of such covers, and provide a few applications to the proof of monotonicity theorems. A more general and formal presentation of these considerations will appear elsewhere ([12], [13], and [14]) but it has seemed appropriate to communicate here some of the simpler aspects in a more immediate and elementary form.

§1. The ordinary derivative. Certain of the properties of the ordinary derivative and the ordinary extreme derivates lead one immediately to a consideration of the following notion.

DEFINITION 1. A collection $\mathcal{C}$ of closed subintervals of an interval $[a,b]$ is said to be an *ordinary full cover* of $[a,b]$ if to each x in $[a,b]$ there is a number $\delta(x) > 0$ such that every interval I with $x \in I$ and $|I| < \delta(x)$ belongs to $\mathcal{C}$.

Such a cover arises in a number of obvious ways: for example if $\underline{f}'$ is the lower extreme derivate of a function f on an interval [a,b] then the collection $\{I : f(I) > (\underline{f}'(x) - \epsilon)|I|, x \in I\}$ would necessarily be a full cover of [a,b] for any $\epsilon > 0$. The partitioning property available for such covers is immediate and easy to prove.

LEMMA 1.1 <u>If $\mathcal{C}$ is an ordinary full cover of an interval</u> [a,b] <u>then $\mathcal{C}$ contains a partition of every subinterval of</u> [a,b].

PROOF. The argument follows a familiar compactness theme. If $\mathcal{C}$ fails to contain a partition of some interval J then by repeated bisection of J there must be a sequence $\{J_n\}$ of subintervals of J with $|J_n| \to 0$ and $J_{n+1} \subseteq J_n$, with the property that $\mathcal{C}$ contains no partition of any interval J_n. Let x_0 be in the intersection of the sequence $\{J_n\}$ and let $\delta(x_0)$ be that positive number promised in definition 1. Then for large enough n the intervals J_n have $|J_n| < \delta(x_0)$ and $x_0 \in J_n$. That means that $\mathcal{C}$ contains these intervals and so trivially partitions them. This contradiction establishes the lemma.

We move now to a few simple consequences of this lemma.

(1.2) <u>If $\underline{f}'(x) \geq 0$ everywhere on an interval</u> [a,b] <u>then</u> f <u>is nondecreasing on that interval</u>.

To prove this we merely collect for any $\epsilon > 0$ those intervals I for which $f(I) > -\epsilon|I|$. If J is any subinterval of [a,b] then this collection contains a partition $\{I_1, I_2, \ldots, I_n\}$ of J giving $f(J) = \Sigma f(I_i) > -\Sigma\epsilon|I_i| = -\epsilon|J|$. As $\epsilon > 0$ is arbitrary it follows that $f(J) \geq 0$ for any subinterval of [a,b] and hence f is nondecreasing as required.

(1.3) <u>If $\underline{f}'(x) \geq 0$ a.e. and $\underline{f}'(x) > -\infty$ everywhere in an interval</u> [a,b] <u>then</u> f <u>is nondecreasing on that interval.</u>

Let $X_0 = \{x \in [a,b] : \underline{f}'(x) < 0\}$. Then since $|X_0| = 0$ there is for any $\epsilon > 0$ a sequence of open sets $\{G_n\}$ such that $G_n \supseteq X_0$ and $|G_n| < \epsilon/n2^n$. Define the following collections of intervals:

$$\mathcal{C}_0 = \{I : f(I) > -\epsilon|I|\} \qquad \text{and}$$

$$\mathcal{C}_n = \{I : f(I) > -n|I| , \; I \subseteq G_n\}$$

for $n = 1,2,3, \ldots$.

If we set $\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$ then it is easy to check that $\mathcal{C}$ is an ordinary full cover of [a,b] . Let J be an arbitrary subinterval of [a,b] and let $\{I_1, I_2, I_3, \ldots, I_m\}$ be a partition of J contained in $\mathcal{C}$. We may compute the sum $\Sigma_{i=1}^{m} f(I_i)$ by considering separately the sums Σ_n for intervals I_i from $\mathcal{C}_n$ if there are such. This gives

$$\begin{aligned} f(J) = \Sigma_{i=1}^{m} f(I_i) &= \Sigma_{n=0}^{\infty} [\ \Sigma_n f(I_i)\ ] \\ &> -\epsilon\ \Sigma_0 |I_i| - \Sigma_{n=1}^{\infty} [\ \Sigma_n n|I_i|] \\ &> -\epsilon |J| - \Sigma_{n=1}^{\infty}\ n(\epsilon/n2^n) \\ &> -\epsilon |J| - \epsilon\ . \end{aligned}$$

Again as $\epsilon > 0$ is arbitrary this gives $f(J) \geq 0$ and the result follows.

Further generalization is possible in (1.3) by permitting an exceptional set in the $\underline{f}'(x) > -\infty$ requirement and adding hypotheses on f. A proof only requires adding more intervals to $\mathcal{C}$ so as to have again an ordinary full cover.

§2. <u>The approximate derivative</u>. If f is a measurable function on an interval $[a,b]$ then we will write $\underline{f}'_{ap}$ for the lower approximate derivate of f. A completely analagous theory for this derivative is available. The notion of an approximate full cover can be developed and a partitioning property proved; applications are then immediate for the proofs would be unchanged.

DEFINITION 2. A collection $\mathcal{C}$ of closed subintervals of an interval $[a,b]$ is said to be an <u>approximate full cover</u> of $[a,b]$ if to each x in $[a,b]$ there is a measurable set $A_x \subseteq [a,b]$ that has left and right density 1 at x such that every interval I with $x \in I$ and with endpoints in A_x belongs to $\mathcal{C}$.

Note that any ordinary full cover is also an approximate full cover but not conversely, and that this latter concept would play the same role in the study of the approximate derivates as the former plays in the study of the ordinary derivates. The partitioning property is identical but this time the proof lies much deeper and cannot be obtained by a simple compactness argument.

LEMMA 2.1 _If_ $\mathcal{C}$ _is an approximate full cover of an interval_ $[a,b]$ _then_ $\mathcal{C}$ _contains a partition of every subinterval of_ $[a,b]$.

PROOF. We follow a familiar category argument. Let $\{A_x : x \in [a,b]\}$ be given as in definition 2 for the collection $\mathcal{C}$ and write

$$E_n = \{x \in [a,b] : |A_x \cap [x,x+t]| > t/2, \ |A_x \cap [x-t,x]| > t/2, \quad \text{for all } 0 < t < 1/n\} .$$

We must then have $\bigcup_{n=1}^{\infty} E_n = [a,b]$. Let $\mathcal{I}$ be the collection of all subintervals of $[a,b]$ such that $\mathcal{C}$ contains a partition of every further subinterval; write $G = \cup\{\text{int } I : I \in \mathcal{I}\}$ and $F = [a,b] \setminus G$. It is easy to check that F is closed, that every interval complementary to F belongs to $\mathcal{I}$, and that (therefore) F is either perfect or empty. The lemma is proved if we are able to show that $F = \emptyset$.

Suppose contrary to the lemma that F is nonempty; then by Baire's theorem (Saks [11,p.54]) there is a nonempty portion $(c,d)\cap F$ such that some set E_n is dense in $(c,d)\cap F$. We may assume that $d - c < 1/n$. We shall show then that $[c,d]$ belongs to $\mathcal{J}$ and since this is impossible the lemma is proved. Let $[\xi, \eta]$ be a subinterval of $[c,d]$. If (ξ, η) contains no points of F then certainly there is a partition of $[\xi, \eta]$; on the other hand if $(\xi,\eta)\cap F \neq \emptyset$ we may write $\xi_1 = \inf (\xi,\eta)\cap F$ and $\eta_1 = \sup (\xi,\eta)\cap F$. As E_n is dense in $F\cap(c,d)$ there are points of E_n in (ξ_1,η_1) arbitrarily close to ξ_1 and η_1.

Let $\delta > 0$ be chosen so that if $0 < t < \delta$ then

$$|A_{\xi_1}\cap[\xi_1,\xi_1+t]| > t/2 \quad \text{and} \quad |A_{\eta_1}\cap[\eta_1-t,\eta_1]| > t/2$$

and then select ξ_2 and η_2 ao that

$$\xi_2 \in (\xi_1,\xi_1+\delta)\cap E_n \quad \text{and} \quad \eta_2 \in (\eta_1 - \delta,\eta_1)\cap F_n .$$

We claim now that $\mathcal{C}$ contains a partition of each of the following (possibly degenerate) intervals: $[\xi,\xi_1]$, $[\xi_1,\xi_2]$, $[\xi_2,\eta_2]$, $[\eta_2,\eta_1]$, and $[\eta_1,\eta]$. This means that $\mathcal{C}$ contains a partition of $[\xi,\eta]$ and the lemma is proved. As $[\xi,\xi_1]$ and $[\eta_1,\eta]$ contain in their interiors no point of F there must be a partition of each. For the interval $[\xi_2, \eta_2]$ notice that

$|A_{\xi_2} \cap [\xi_2,\eta_2]| > (\eta_2-\xi_2)/2$ and

$|A_{\eta_2} \cap [\xi_2,\eta_2]| > (\eta_2-\xi_2)/2$ so that there must exist a point z in both of these sets and this requires that $[\xi_2,z]$ and $[z,\eta_2]$ both belong to $\mathcal{C}$; this gives a partition of $[\xi_2,\eta_2]$. Identical arguments provide partitions of the other intervals $[\xi_1,\xi_2]$ and thus the lemma is proved.

From this lemma we deduce with proofs identical to those of the previous section several known monotonicity results. The standard proofs may be compared; Bruckner [1,p.156] and Goffman and Neugebauer [5] prove the first of these (but using an exact approximate derivative in place of an extreme derivate) and O'Malley [9] obtains the second as an application of his theory of selective derivates.

(2.2) If $\underline{f}'_{ap}(x) \geq 0$ everywhere on an interval $[a,b]$ then f is nondecreasing on that interval.

(2.3) If $\underline{f}'_{ap}(x) \geq 0$ a.e. and $\underline{f}'_{ap}(x) > -\infty$ everywhere on an interval $[a,b]$ then f is nondecreasing on that interval.

One can extend (2.3) to allow a countable exceptional set. If $\underline{f}'_{ap}(x) \geq 0$ a.e. and $\underline{f}'_{ap}(x) > -\infty$ except possibly in a countable set C then provided one knows that at each point of C

$$\operatorname{ap-}\overline{\lim_{t\to x-}} f(t) \leq \operatorname{ap-}\underline{\lim_{t\to x+}} f(t)$$

the function f must be nondecreasing. The proof requires only a modification in the approximate full covers needed in the original proof.

All of the preceeding can be generalized by replacing the approximate derivative by Denjoy's preponderant derivative (see Bruckner [1,p.165]) and by requiring in definition 2 that the sets A_x have only right and left density exceeding 1/2. Note that the partitioning property of Lemma 2.1 needed only this much of the sets A_x. Perhaps more curious is the fact that one could introduce a "lopsided" preponderant derivative: in definition 2 require that the sets A_x have at x a right density exceeding ρ and a left density exceeding λ. If $\rho + \lambda \geq 1$ then again the proof of Lemma 2.1 will apply with a few modifications. Thus if a function has a (ρ,λ)-preponderant lower derivate everywhere nonnegative with $\rho + \lambda \geq 1$ that function must be nondecreasing.

A generalization in another spirit is provided by replacing density considerations with category ones; this yields the qualitative derivative of Marcus (see Bruckner [1,p.166]) and the theory for qualitative full covers and the proof of the partitioning property of such covers follows closely the pattern given here.

§3. <u>Symmetric derivatives</u>. A similar theory may be developed for the symmetric derivative. It is not clear however what might be the best possible statement of partitioning properties for the notion of a symmetric full cover. We give in Lemma 3.1 a partitioning property which is sufficient to derive a number of monotonicity theorems, but it is not intended as a complete picture of the global properties of such covers.

DEFINITION 3. A collection $\mathcal{S}$ of closed subintervals of an interval (a,b) is said to be a <u>symmetric full cover</u> of (a,b) if to each x in (a,b) there is a number $\delta(x) > 0$ such that every interval $[x-t,x+t]$ with $0 < t < \delta(x)$ belongs to $\mathcal{S}$.

LEMMA 3.1 *Let $\mathcal{S}$ be a symmetric full cover of an interval (a,b) and write $c = (a+b)/2$ for the midpoint of that interval. Then there is a set $D \subseteq (c,b)$ with $\overline{(c,b)\setminus D}$ countable such that $\mathcal{S}$ contains a partition of every interval $[c-x,c+x]$ for every $c+x \in D$.*

PROOF. (*cf*. McGrotty [8]) For simplicity of notation we center (a,b) at 0 and consider $\mathcal{S}$ is a symmetric full cover of $(-b,b)$. Let $D = \{x \in (0,b) : \mathcal{S}$ contains a partition of $[-x,x]\}$. Suppose that $\delta(x)$ is given for each $x \in (-b,b)$ and, with no loss in generality, that $\delta(x) = \delta(-x)$ everywhere.

Define $\beta = \sup \{x \in (0,b) : \overline{(0,x)\setminus D}$ is countable$\}$. The lemma is proved by showing that $\beta = b$. Note firstly that D contains the interval $(0, \delta(0))$ so that $\beta \geq \delta(0) > 0$. Suppose now, contrary to the lemma, that $0 < \beta < b$, and consider the interval $(\beta - \delta(\beta), \beta + \delta(\beta))$. Simple arguments show that

$\overline{(\beta - \delta(\beta), \beta)\setminus D}$ is countable and that

$\overline{(\beta, \beta + \delta(\beta))\setminus D}$ is uncountable. But $\mathcal{S}$ contains every interval $[\beta - t, \beta + t]$ and $[-\beta - t, -\beta + t]$ for $0 < t < \delta(\beta)$. From this we can see that D contains in the interval $(\beta, \beta + \delta(\beta))$ at least a reflection of the set $D \cap (\beta - \delta(\beta), \beta)$ about the point β. Consequently $\overline{(\beta, \beta + \delta(\beta))\setminus D}$ must also be countable and this is a contradiction. It follows that $\beta = b$ and the lemma is proved. Note that we have proved as well that for any $x \in D$ there is an $\epsilon, 0 < \epsilon < \delta(0)$, such that $\mathcal{S}$ contains a partition of the interval $[\epsilon, x]$.

Using this partitioning property of symmetric full covers we can prove several monotonicity theorems by identical arguments to those given above. Here $\underline{f}'_{sy}$ will denote the lower extreme symmetric derivate.

(3.2) *If* $\underline{f}'_{sy}(x) > 0$ *everywhere in* (a,b) *then there* *is a set* $D \subseteq (c,b)$, *where* c *is the midpoint of* (a,b), *the closure of whose complement in* (c,b) *is countable such that* $f(c-x) < f(c+x)$ *for every* $c+x$ *in* D .

Also we have:

(3.3) *If* $\underline{f}'_{sy}(x) \geq 0$ *a.e. in* (a,b) *and* $\underline{f}'_{sy}(x) > -\infty$ *everywhere in* (a,b) *then except possibly for a countable set* $f(c-x) \leq f(c+x)$, $0 < x < (b-a)/2$.

The proof is only mildly more complicated than that of (1.3) but requires some attention to the fact that the exceptional countable closed set may vary with the ε . We shall not give the details.

From this we obtain some known monotonicity theorems (*eg*. Weil [15], and Evans [3]).

(3.3) *Suppose that* $\underline{f}'_{sy}(x) \geq 0$ a.e. *in* (a,b) *and everywhere in* (a,b) $\underline{f}'_{sy}(x) > -\infty$. *If*

$$f(a) \leq \underline{\lim}_{x \to a+} f(x) \quad \text{and } f(b) \geq \overline{\lim}_{x \to b-} f(x) ,$$

then $f(a) \leq f(b)$.

(In fact we could use less than these one sided semi-continuity conditions. We only need to know that the sets

$\{x \in (a,b) : f(x) > f(a) - \epsilon\}$ and

$\{x \in (a,b) : f(x) < f(b) + \epsilon\}$ are for any $\epsilon > 0$

uncountable in any neighbourhood of a and b.

We conclude with one further type of partitioning property of symmetric full covers. This is in quite a different spirit than those we have so far discussed.

LEMMA 3.4 *Let $\mathcal{S}$ be a full symmetric cover of an interval* (a,b). *Then there must exist an interval* $(c,d) \subseteq (a,b)$ *such that to every subinterval* J *there are* $\{J_0, J_1, J_2, J_3, J_4, J_5\}$ *each a subinterval of* J *and each belonging to $\mathcal{S}$ such that*

$$f(J) = \Sigma_{i=0}^{5} (-1)^i f(J_i)$$

for any function f.

PROOF. (cf. Davies [2]) Let $\delta(x)$ correspond to the cover as in definition 3 and write $E_n = \{x \in (a,b) : \delta(x) > 1/n\}$. Then since the sets $\{F_n\}$ cover (a,b) there must exist an interval $(c,d) \subseteq (a,b)$ and a set E_n dense in (c,d). Without loss in generality we may assume that $d - c < 1/n$. If $I \subseteq (c,d)$ we must exhibit a sequence of subintervals chosen from $\mathcal{S}$ that has the property stated.

The notation is considerably simplified if we assume that $I = [-2\alpha, 2\alpha]$, that E_n is dense in

$(-2\alpha, 2\alpha)$, and that $4\alpha < 1/n$. If so choose a negative x' in E_n so that $0 < x' - (-\alpha/2) < \delta(\alpha)/2$, and choose a positive x'' in E_n so that $0 < x'' - (\alpha/2) < \delta(-\alpha)/2$. This means that $0 < 2x' + \alpha < \delta(\alpha)$ and $0 < 2x'' - \alpha < \delta(\alpha)$. Clearly we can arrange too that $x'' + 2x' < 0$ and $2x'' + x' > 0$.

Let us define now the following intervals:

$$I_1 = [-2\alpha, 2x'+2\alpha] = [x'-(x'+2\alpha), x'+(x'+2\alpha)] ,$$

$$I_2 = [-2x', 2x'+2\alpha] = [\alpha-(2x'+\alpha), \alpha+(2x'+\alpha)] ,$$

$$I_3 = [2x''+2x', -2x'] = [x''+(x''+2x'), x''-(x''+2x')] ,$$

$$I_4 = [-2x'', 2x'+2x''] = [x'-(2x''+x'), x'+(2x''+x')] ,$$

$$I_5 = [-2x'', 2x''-2\alpha] = [-\alpha-(2x''-\alpha), -\alpha+(2x''-\alpha)] , \text{ and}$$

$$I_6 = [2x''-2\alpha, 2\alpha] = [x''-(2\alpha-x''), x''+(2\alpha-x'')] .$$

It is straightforward to check that each I_i $(i=1,2,\ldots,6)$ is a subinterval of I, that each belongs to $\mathcal{S}$ and that for any function f,

$$f(I) = f(I_1) - f(I_2) - f(I_3) - f(I_4) + f(I_5) + f(I_6) .$$

It remains only to relabel as in the lemma and the proof is complete.

As an application note that Lemmas (3.3) and (3.4) provide another proof of an observation that has been made in this Exchange on the subject of symmetric functions. See Davies [2], Rusza [10], Foran [4], and the editors [16] . A function is symmetric if there is a symmetric full cover of $(-\infty,+\infty)$, $\mathcal{S}$ for which $f(I) = 0$ for every $I \in \mathcal{S}$. By (3.4) we see immediately that there must be an interval (α,β) on which f is constant (in fact of course such intervals are everywhere dense). Then using (3.3) we can find a countable closed set C such that $\mathcal{S}$ contains a partition of $[\gamma,x]$ for every $x \notin C$ and some $\gamma \in (\alpha,\beta)$. Consequently f must be constant in the complement of C.

REFERENCES

[1] A.M. Bruckner, Differentiation of real functions. Lecture Notes in Math. #659, Springer Verlag (1978).

[2] R.O. Davies, Symmetric sets are measurable, Real Analysis Exchange 4 (1978/79). 87-89.

[3] M.J. Evans, A symmetric condition for monotonicity, Real Analysis Exchange 3 (1977/78), 98-99.

[4] M. Foran, Symmetric functions, Real Analysis Exchange 1 (1976), 38-40.

[5] C. Goffman and C. Neugebauer, On approximate derivatives. Proc. Amer. Math. Soc., 11 (1960), 962-966.

[6] E. Goursat, Sur la definition générale des fonctions analytique, d'après Cauchy, Trans, Amer. Math. Soc., 1 (1900), 14-16.

[7] R. Henstock, Linear Analysis, Butterworths, London (1968).

[8] J. McGrotty, A theorem on complete sets, J. London Math. Soc., 37 (1962), 338-340.

[9] R. O'Malley, Selective derivates, Acta. Math. Acad. Sci. Hungar., 29 (1977), 77-97.

[10] I.Z. Ruzsa, Locally symmetric functions, Real Analysis Exchange 4 (1978/79), 84-86.

[11] S. Saks, Theory of the integral, Warsaw (1937).

[12] B.S. Thomson, Some monotonicity theorems, (submitted).

[13] ______________, On the Goldowski-Tonelli theorem, (submitted).

[14] ______________, Generalized extreme derivates, (in preparation).

[15] C.Weil, Monontonicity, convexity and symmetric derivatives, Trans. Amer. Math. Soc., 222 (1976), 225-237.

[16] Editorial staff, Concerning query 37, Real Analysis Exchange, 4 (1978/79), 82-83.

Received August 1, 1980