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ON LOCALLY SYMMETRIC AND SYMMETRICALLY CONTINUOUS FUNCTIONS

Introduction

The function $f: R \to R$ is said to be <u>locally symmetric at the point</u> $x \in R$ if there exists a $\delta = \delta(x)$ such that for each h, $0<h<\delta$, the equality $f(x - h) = f(x + h)$ holds. The function $f: R \to R$ is said to be <u>locally symmetric</u> (on R) if it is locally symmetric at each point $x \in R$ (cf. [1], [2], [4]).

The function $f: R \to R$ is said to be <u>symmetrically continuous at the point</u> $x \in R$ if $\lim_{h\to 0} (f(x + h) - f(x - h))=0$. The function $f: R \to R$ is said to be <u>symmetrically continuous</u> (on R) if it is symmetrically continuous at each point $x \in R$ (cf. [1], [3]).

Denote by LS and SC the class of all locally symmetric functions and all symmetrically continuous functions, respectively. Clearly we have

(1) $$LS \subset SC .$$

In the papers [1], [2], [4] some results about the functions of the class LS are proved. In connection with these results we shall prove some further results about functions of the classes LS and SC; in so doing, we shall make use of the following theorem of T.Z. Ruzsa[4].

THEOREM R. _If_ $f \in LS$, _then for a suitable_ $q \in R$ _the closure of the set_ $\{x \in R: f(x) \neq q\}$ _is countable_. In the first part of this paper we shall study the classes LS and SC from the viewpoint of pointwise and uniform convergence of sequences of functions from these classes and from the viewpoint of the convergence of transfinite sequences of functions from these classes.

In the second part of the paper we shall study the relationship between the linear normed space of all bounded symmetrically continuous functions and its subspace of all bounded locally symmetric functions.

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At first we shall establish the relationship of locally symmetric functions to Baire functions.

Theorem 1.1 _Each locally symmetric function_ $f: R \to R$ _belongs to the first Baire class_.

Proof. Set $Q=\{x: f(x)=q\}$, where q is the number guaranteed by Theorem R. If $M^a=\{x: f(x)<a\}$, then for $a \leq q$ M^a is a countable set and for $a > q$, M^a is the union of int(Q) and a countable set. In either case, M^a is an F_σ-set and the proof is complete.

We shall investigate the uniform closure of the classes LS and SC.

Theorem 1.2. *If* $f_n \in SC(n=1,2,\ldots)$ *and the sequence* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly* (on R) *to the function* $f: R \to R$, *then* $f \in SC$.

Proof. Let $x \in R$. We shall show that

(2) $$\lim_{h \to 0} (f(x + h) - f(x - h)) = 0.$$

Let $\epsilon > 0$. By hypothesis there exists an m such that for each $t \in R$ we have

(3) $$|f_m(t) - f(t)| < \frac{\epsilon}{3} .$$

Since $f_m \in SC$, there is a $\delta > 0$ such that for each $h \in (0, \delta)$ we have

(4) $$|f_m(x + h) - f_m(x - h)| < \frac{\epsilon}{3} .$$

For $0 < h < \delta$ we have $|f(x+h)-f(x-h)| \leq |f(x+h)\ f_m(x+h)| + |f_m(x+h)-f_m(x-h)| + |f_m(x-h)-f(x-h)|$, where the first and third summand on the right-hand sideare less than $\frac{\epsilon}{3}$ because of (3), and the second is less than $\frac{\epsilon}{3}$ because of (4). From this (2) follows at once.

Hence the class SC is closed with respect to the uniform convergence. The class LS does not have the similar property.

Theorem 1 3. *There exists a sequence* $\{f_n\}_1^{\infty}$, $f_n \in LS$ $(n=1\ 2,\ldots)$ *which converges uniformly on* R *to a function* f *that is locally symmetric at no irrational point*.

Proof. Let $p_1 < p_2 < \ldots$ be the sequence of all prime numbers. Put $A_n = \{\frac{k}{p_n} : k$ is integer, $p_n \nmid k\}$ $(n=1,2 \ldots)$. Then evidently $A_n \cap A_m = \emptyset$ for $n \neq m$. Define the function $f_n (n=1,2 \ldots)$ in the following way:

$f_n(x)=i^{-1}$ for $x \in A_i$ $(i=1,2,\ldots,n)$ and $f_n(x)=0$ for $x \in R - \bigcup_{i=1}^{n} A_i$.

Then evidently $f_n \in LS$ $(n=1,2,\ldots)$. The sequence $\{f_n\}_1^\infty$ converges uniformly to the limit function f given by $f(x)=i^{-1}$ for $x \in A_i$ $(i=1,2,\ldots)$ and $f(x)=0$ for $x \in R - \bigcup_{i=1}^{\infty} A_i$.

Let x be an irrational point, and let $\delta>0$. Then there exist two positive integers n and k such that $p_n \nmid k$ and $x-\delta<\frac{k}{p_n}<x$. Since $\frac{1}{n} = f(\frac{k}{p_n}) \neq f(x+(x-\frac{k}{p_n}))=0$, f is not locally symmetric at the point x. This ends the proof.

If $\{f_n\}_1^\infty$ is a pointwise convergent sequence of functions from the class LS , then the limit function f belongs, on account of Theorem 1.1, to the second Baire class. Moreover we shall prove now that f is an "honorary function in the second Baire class", i.e., we get from f a function belonging to the first Baire class if we change the values of f at the points of a countable set.

Theorem 1.4. The limit function of a pointwise convergent sequence of locally symmetric functions is constant on a co-countable set; furthermore if the *convergence is uniform on* R, *then there exists a* $q\in R$ *such that for each* $\epsilon>0$, *the closure of the set* $V_\epsilon=\{x: |f(x)-q|>\epsilon\}$ *is countable*.

Proof. Let $f_n\in LS$ $(n=1,2,\ldots)$, $f_n\to f$. We already know that for each n there exists a $q_n\in R$ and a countable set $B_n\subset R$ such that for each $x\in R-B_n$ we have $f_n(x)=q_n$ (see Theorem R.). But then for each $x\in R-\bigcup_{n=1}^{\infty} B_n$ we have $f_n(x)=q_n\to f(x)$ hence $f(x)=q=\lim_{n\to\infty} q_n$ and the set $\bigcup_{n=1}^{\infty} B_n$ is countable.

Now assume that the convergence is uniform on R. Choose an m such that for each $x\in R$ we have

$$|f_m(x) - f(x)| < \frac{\epsilon}{3}$$

(especially $|q_m-q|<\frac{\epsilon}{3}$).

Put $W_\epsilon=\{x\in R: |f_m(x)-q_m|>\frac{\epsilon}{3}\}$. According to Theorem R., the closure of the set W_ϵ is countable. Further it can be easily verified that $V_\epsilon\subset W_\epsilon$. Therefore the closure of the set V_ϵ is countable, too, and the theorem is proved.

Remark 1.1 Let us observe that the function f from the foregoing theorem need not belong to the first Baire class. It suffices to put (the notation being the same as in the proof of Theorem 1.3): $f_n(x)=i^{-1}$ for $x\in A_i$ $(i=1,2,\ldots,n)$ and $f_n(x)=2$ for $x\notin\bigcup_{i=1}^{n} A_i$. Then

$f_n \to f$, $f(x)=i^{-1}$ for $x \in A_i$ $(i=1,2,\ldots)$ and $f(x)=2$ for $x \in R - \bigcup_{i=1}^{\infty} A_i$. Since the set $\bigcup_{i=1}^{\infty} A_i$ is dense in R (see the definition of sets A_i in the proof of Theorem 1.3.), the function f is discontinuous at each point of R and therefore it is not in the first Baire class.

We shall now prove that the classes LS and SC are closed with respect to transfinite convergence (see [5]).

At first we introduce the following simple auxiliary result, the proof of which can be left to the reader.

Lemma 1.1. (i) The function $f\colon R \to R$ is locally symmetric at the point $x \in R$ if and only if for two arbitrary sequences $\{x_n\}_1^{\infty}$, $\{y_n\}_1^{\infty}$, of real numbers with $x_n \uparrow x$, $y_n \downarrow x$, $|x_n - x| = |y_n - x|$ $(n = 1,2,\ldots)$ there exists a p such that for each $n > p$ we have $f(x_n) = f(y_n)$.

(ii) The function $f\colon R \to R$ is symmetrically continuous at the point $x \in R$ if and only if for two arbitrary sequences $\{x_n\}_1^{\infty}$, $\{y_n\}_1^{\infty}$ of real numbers with $x_n \uparrow x, y_n \downarrow x, |x_n - x| = |y_n - x|$ $(n = 1,2,\ldots)$ we have

$$\lim_{n\to\infty} (f(x_n) - f(y_n)) = 0$$

Theorem 1.5. Let $\{f_\xi\}_{\xi<\Omega}$ (Ω is the first uncountable ordinal number) be a transfinite sequence of functions from SC (LS), let $f_\xi \to f$. Then $f \in SC$ $(f \in LS)$.

Proof. We shall give the proof only for functions $f_\xi \in LS$ (the second case can be proved in an analogous way).

Let $x \in R$ and let $\{x_n\}_1^\infty$, $\{y_n\}_1^\infty$ be two sequences of real numbers with $x_n \uparrow x$, $y_n \downarrow x$, $|x_n - x| = |y_n - x|$ $(n=1,2,\dots)$. Since $f_\xi \to f$, to each $n=1,2,\dots$ there exist ordinal numbers α_n, β_n such that

$$(5) \qquad f_\xi(x_n) = f(x_n) \text{ for each } \xi > \alpha_n,$$

$$f_\xi(y_n) = f(y_n) \text{ for each } \xi > \beta_n.$$

(See [5]). Let α be the first ordinal number which follows after all α_n, β_n $(n=1,2,\dots)$. Then $\alpha < \Omega$ and hence for each $n=1,2$ ... we get from (5) that

$$(6) \qquad f_\alpha(x_n) = f(x_n),\ f_\alpha(y_n) = f(y_n).$$

Since $f_\alpha \in LS$, there exists, on account of Lemma 1.1, a p such that for each $n>p$ we have $f_\alpha(x_n) = f_\alpha(y_n)$. But then for $n>p$ we have according to (6) the equality $f(x_n) = f(y_n)$. Now Lemma 1.1 guarantees that f is locally symmetric at the point x, and the proof is finished.

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Denote by $(M, \|\cdot\|)$ the linear normed space of all bounded functions $f\colon R \to R$ with the norm $\|f\| = \sup_{t \in R}\{|f(t)|\}$. Denote by LS* and SC* the class of all $f \in M$ that are locally symmetric on R and symmetrically continuous on R, respectively.

It follows from Theorem 1.2 that SC* is a closed subspace of M. Since evidently $SC^* \neq M$, the set SC^* is nowhere dense in M.

It follows from (1) that LS^* is a linear subspace of the space SC^*, but LS^* is not a closed set in SC^* (see Theorem 1.3). The question arises whether the set LS^* is a nowhere dense set in SC^*. The answer is affirmative.

Theorem 2.1. The set LS^* is nowhere dense in SC^*.

Proof Let $K(f,\delta)$ be a sphere in SC^* ($f \in SC^*, \sigma > 0$). We shall prove that there exists a sphere $K' \subset K(f,\delta)$ in SC^* such that $K' \cap LS^* = \emptyset$.

If $LS^* \cap K(f,\delta) = \emptyset$, we can take $K' = K(f,\delta)$. Let there exist $g \in LS^* \cap K(f,\delta)$. We have already seen that there exists a number $q \in R$ and an interval $[a',b'] \subset R$ such that $g(x)=q$ for each $x \in [a',b']$. Choose a $\delta' > 0$ such that $K(g,\delta') \subset K(f,\delta)$. Define the function $h\colon R \to R$ in the following way. We choose some numbers a, a'', b, b'' with $a' < a < a'' < b'' < b < b'$ and put

$h(x) = q - \frac{\delta'}{2}$ for $a \le x \le a''$, $h(x) = q + \frac{\delta'}{2}$ for $b'' \le x \le b$; further let h be a linear and continuous function on each of the intervals $[a'',b'']$, $[a',a]$, $[b,b']$ and $h(a')=g(a')$, $h(b') = g(b')$. Let $h(x) = g(x)$ for $x \notin [a', b']$. Then clearly $h \in SC^*$ and

$$(7) \qquad \|h - g\| \le \frac{\delta'}{2}.$$

Now $K(h,\frac{\delta'}{3}) \subset K(g,\delta')$ since for $v \in K(h\,\frac{\delta'}{3})$ we have according to (7) $\|v - g\| \le \|v - h\| + \|h - g\| \le \frac{\delta'}{3} + \frac{\delta'}{2} < \delta'$. We shall prove that

(8) $$LS^* \cap K(h, \frac{\delta'}{3}) = \emptyset \ .$$

If $v \subset K(h,\frac{\delta'}{3})$, then for each $x \in [a,a'']$ we have $v(x) < h(x)+ \frac{\delta'}{3} = q - \frac{\delta'}{2} + \frac{\delta'}{3} = q - \frac{\delta'}{6}$ and for each $x \in [b'',b]$ we have $v(x) > h(x) - \frac{\delta'}{3} = q + \frac{\delta'}{2} - \frac{\delta'}{3} = q + \frac{\delta'}{6}$. All the values of the function v on $[a,a'']$ are less than the values on $[b'',b]$. Therefore the function v does not belong to LS^* because of Theorem R. Hence (8) is established and this ends the proof.

The following theorem, which is an immediate consequence of Theorems 1.2 and 1.4, gives some properties of functions belonging to the closure (in SC^*) $\overline{LS}^*$ of the class LS^* .

Theorem 2.2. <u>If</u> $f \in \overline{LS}^*$, <u>then</u> f <u>is bounded, symmetrically continuous, and there exists a</u> $q \in R$ <u>such that for each</u> $\epsilon > 0$ <u>the closure of the set</u> $V_\epsilon = \{x \in R: \ |f(x) - q| > \epsilon\}$ <u>is countable</u>.

Remark 2.1. The question about characterising the functions from $\overline{LS}^*$ remains open.

References

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