APPROXIMATE DIFFERENTIATION

by

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Introduction.

Among the many notions of generalized differentiation that mathematicians have studied during the last century, perhaps the notion of approximate differentiation has received the most attention. This may be so because the approximate derivative (or differential in the case of several variables) can often serve as a substitute for the ordinary derivative (or total differential) when the latter does not exist. It is also so partly because some of the concepts, theorems and techniques that arise in studies involving approximate differentiation have applications to questions involving approximations of functions of one or more variables, surface area, integration theory and, of course, differentiation theory in general. These various uses of approximate differentiation can be viewed as part of the "almost everywhere" theory of approximate differentiation. During the last twenty years or so there has also been considerable interest in the "everywhere" theory. This recent interest is a consequence of two facts: i) the behavior of the ordinary derivative (assumed to exist everywhere) is not fully understood and ii) the behavior of the approximate derivative seems to mimic that of the ordinary derivative.

Our present purpose is two-fold. In Chapters I, II and III we discuss some of the desirable aspects of approximate differentiation as well as some of the ways in which approximate differentiation and the circle of ideas surrounding it interplay with other parts of analysis. All of the chapters focus on the "almost everywhere" part of the theory. Then in Chapter IV we turn to the "everywhere" part of the theory in one variable, compare the approximate derivative with the ordinary derivative, and indicate the nature of some of the more important open problems.

We mention that while most of what we discuss in the short Chapter I is classical (and much can be found in Saks' book [60]), the bulk of the material of the other chapters is more recent and most of it is not readily available in book form.

I. Stepanov's Theorem.

Our first aim is to give some indication of the advantages of approximate differentiation over ordinary differentiation and why such advantages exist. To fix ideas, we begin with some definitions which will be used throughout the paper. Fundamental to any "approximate" notion is the concept of density of a set at a point.

<u>Definition</u>. Let S be a measurable subset of R^n. A point $x \in R^n$ is called a point of density of S if $\lim_{\delta(I) \to 0} \frac{\lambda(S \cap I)}{\lambda(I)} = 1$. Here λ denotes n dimensional Lebesgue measure and $\delta(I)$ denotes the diameter of the n dimensional

cube centered at x and having edges parallel to the coordinate axes (i.e. oriented cubes).

One finds several variants of the above definition in the literature. The definition we have chosen is adequate for most of our purposes. We shall, however, have to consider variants in Chapter III and IV and will indicate the necessary modifications at that time.

Definition. Let F be a real valued function defined on R^n and let $x \in R^n$. The function F is said to be approximately continuous at x provided there is a set S, having x as a point of density, such that the restriction of F to S is continuous at x. Similarly, F is said to be approximately differentiable at x if there exists a set S, having x as a point of density, such that the differential of F at x exists with respect to S. Notions such as approximate partials and approximate partial derivates can now be defined in the obvious manner.

A basic fact is that if S is measurable, then almost every point of S is a point of density of S. Furthermore, almost every point of S is a point of linear density of S in all n coordinate directions.

A first advantage of the "approximate" notion is that the approximate partial derivates of F reflect the measurability properties of F while the ordinary partial derivates do not. Thus, if F is Borel or Lebesgue measurable, so are $D^+_{ap}F_i$, $i = 1,\ldots,n$ and the other approximate derivates, but their ordinary counterparts need not be (unless $n = 1$). This may seem surprising at first because approximate differentiation is intrinsically more complex than is ordinary differentiation. But this intrinsic complexity allows one to ignore the behavior of F on certain sets which can adversely affect the bahavior of the partial derivates. Consider the example of Neubauer [49]. Let A be a nonmeasurable subset of R^1 and let Z denote the set of rational numbers. The set $B = Z \times A$ has planar measure zero, so the characteristic function F of B is measurable. Let $T = R \times \tilde{A}$ (where $\tilde{A}$ denotes the complement of A), $U = \tilde{Z} \times A$. Then $D^+F_1 = 0$ on T while $D^+F_1 = \infty$ on U. It follows that D^+F_1 is nonmeasurable. Note that $D^+_{ap}F_1 = 0$ almost everywhere and is therefore measurable. It is precisely the fact that one can ignore the bahavior of F on certain small linear sets in computing the approximate partials that renders $D^+_{ap}F_1$ measurable.

We mention in passing that not all measurability properties of F can be lost by the partial derivates: if F is continuous, these derivates are Borel measurable, and if F is Borel measurable, the derivates are Lebesgue measurable, indeed, analytic in the sense of Lusin.

The good behavior of approximate derivatives as compared with ordinary derivatives becomes even more pronounced in the case of total differentials. For a function of 2 variables, it is possible for the partial derivatives to exist almost everywhere but for the total differential to exist nowhere. Examples of this fact may be very simple. Let Q be the closed unit square and let $S = \{x_1, x_2, \ldots\}$ be a countable dense set in the interior of Q. Let $\{\epsilon_n\}$ and $\{k_n\}$ be two sequences of positive numbers such that

$\sum_n \epsilon_n < \infty$, $\lim_n k_n = \infty$ and the disk D_n centered at x_n

having radius ϵ_n is contained in Q. For each positive integer n, let F_n be the continuous function which vanishes off D_n and whose graph on D_n is a right circular cone of altitude k_n. Let $F = \sum_n F_n$. We observe first that F is defined on almost all lines parallel to the coordinate axes (and intersecting Q) and is absolutely continuous in each variable for almost all values of the other variable. To see this, let I_n be the projection of D_n on the x axis. Since $\delta(D_n) = 2\epsilon_n$, $\sum_n \delta(D_n) < \infty$. It follows that almost every $x \in [0,1]$ is in only finitely many of the sets I_n. Thus, almost every vertical line intersects only finitely many of the disks D_n and $F(x,\cdot)$ is absolutely continuous on each such line. It follows that the partial with respect to y exists almost everywhere. The same is true of the other partial.

On the other hand, F is nowhere totally differentiable. To see this let D be any disk contained in Q. Since infinitely many points of S will lie in D and $\lim_n k_n = \infty$, it follows that F is unbounded in D. Thus F is nowhere continuous, and a fortiori nowhere differentiable.

More delicate examples exist for which F is continuous and possesses the other properties of the example above (Rademacher [59]).

As one can anticipate from the preceding discussion, this type of situation cannot arise in the "approximate" setting. In fact we have Stepanov's theorem:

Theorem (Stepanov [60]). Let F be measurable on R^n. Then F is approximately differentiable almost everywhere on R^n if and only if each of the approximate partial derivatives exists almost everywhere on R^n.

II. Lusin type theorems.

Lusin's basic theorem in measure theory asserts that a measurable function can be approximated in the Lusin sense by a continuous function. It seems natural to expect that by a suitable narrowing of the class of measurable functions, one can obtain more delicate approximations of the Lusin type. We discuss such approximations first in the setting of R^1 and then discuss the analogues in R^n, $n > 1$. We shall see that appropriate Lusin type approximations can be used to actually characterize important function classes.

1. Approximation for functions of one variable.

We begin with the observation [11, p. 19] that a function is measurable if and only if it is approximately continuous almost everywhere. Thus, Lusin's theorem can take the following form.

Lusin's Theorem. A function F defined on an interval I is approximately continuous almost everywhere if and only if to each $\epsilon > 0$ corresponds a closed set K and a continuous function G such that $\lambda(I \sim K) < \epsilon$ and $F = G$ on K.

What happens if we require more of F, namely that F be approximately differentiable almost everywhere? Can we take G to be differentiable in the conclusion of Lusin's theorem? The following theorem of Whitney asserts that we can do even better.

Whitney's Theorem [71]. A function F defined on an interval I is approximately differentiable almost everywhere if and only if to each $\epsilon > 0$ corresponds a closed set K and a continuously differentiable function G such that $\lambda(I \sim K) < \epsilon$ and $F = G$ on K.

The two theorems we have stated give characterizations of two classes of functions in terms of Lusin-type approximations. But each of these classes is defined in terms of some sort of "approximate" notion. It is, perhaps, surprising that Lusin-type approximations can be used to characterize certain important classes of functions whose definitions do not involve any "approximate" notions. The next three theorems are of this type, and illustrate how the approximation improves as the class narrows.

Theorem (Michael [44]). A function F defined on an interval I is equivalent to a function of bounded variation if and only if to each $\epsilon > 0$ corresponds a closed set K and a continuously differentiable function G such that $\lambda(I \sim K) < \epsilon$, $F = G$ on K, $|\mu_F(I) - \mu_G(I)| < \epsilon$.

Here μ_F and μ_G denote the infimums of the variation measures of functions equivalent to F and G respectively.

With absolute continuity replacing bounded variation the approximation improves.

Theorem (Michael [44]). A function F defined on an interval I is equivalent to an absolutely continuous function if and only if to each $\epsilon > 0$ corresponds a closed set K and a continuously differentiable G such that $\lambda(I \sim K) < \epsilon$, $F = G$ on K, $\mu_F(I \sim K) < \epsilon$ and $\mu_G(I \sim K) < \epsilon$.

Here $\mu_F(I \sim K)$, for example, denotes the sum of μ_F on the sequence of intervals contiguous to K.

In the two preceding theorems, the class of approximating functions was the same. It was the method of approximation that improved. More recently, Liu [39] considered the intermediate class of continuous functions of bounded variation. Here, the method of approximation is the better one (which applies to the absolutely continuous case), but the class of approximating functions must naturally be a bit larger.

Theorem. (Liu [39]). A function F defined on an interval I is equivalent to a continuous function of bounded variation if and only if to each $\epsilon > 0$ corresponds a closed set K and a function G with continuously turning tangent such that $\lambda(I \sim K) < \epsilon$, $F = G$ on K, $\mu_F(I \sim K) < \epsilon$, and $\mu_G(I \sim K) < \epsilon$.

Thus, we can no longer require that G be continuously differentiable. This requirement has been weakened to the condition that G' be continuous in the extended sense, i.e. $\lim_{t \to x} G'(t) = G'(x)$ for all x, (infinite values allowed).

The proofs of the preceding three theorems all follow the same lines. One first uses Whitney's theorem to obtain an appropriate closed set K and continuously differentiable G. One then extends the restriction of G to K in such a way as to reduce the variation measure of G as needed while preserving the necessary differentiability properties of G. In the case of Liu's theorem, one must, of course, lose a bit of the differentiability structure of G.

We observe that in all three of the preceding theorems we can replace Lebesgue measure and the variation measure by a single "length" measure. (c.f. Chapter II, Section 2).

We mention in closing this section that some similar interesting results have recently been obtained by Garg [17].

2. <u>Analogues for functions of several variables</u>.

The one dimensional results above have analogues for functions of n variables. Since measurable functions are approximately continuous almost everywhere, Lusin's theorem may again be stated in the form that if F is approximately continuous almost everywhere then to each $\epsilon > 0$ corresponds a continuous G such that $F = G$ except on a set of measure less than ϵ.

The analogue of Whitney's approximation theorem requires the Whitney Extension Theorem [70]. If $S \subset R^n$

is closed, a function $F:S \to R^1$ is said to be of class C^1 on S if for each $x = (x_1,\ldots,x_n) \in S$, there is a vector $a = (a_1,\ldots,a_n)$, such that for every $\epsilon > 0$ there is a $\delta > 0$ such that if y and z are both at distance less that δ from x, i.e. $|y - x| < \delta$ and $|z - x| < \delta$, then $|F(y) - F(z) - \sum_{i=1}^{n} a_i(y_i - z_i)| < \epsilon |y-z|$.

This condition is clearly necessary in order that F may be extended to a function of class C^1 on R^n. The Whitney Extension Theorem asserts that it is also sufficient. Thus, a function F defined on a closed set in R^n may be extended to a function of class C^1 on all of R^n if and only if it is of class C^1 on the closed set.

Now, suppose F is defined on the unit cube Q and is approximately differentiable almost everywhere. For each $\epsilon > 0$ it is possible to find a closed $S \subset Q$, of measure greater than $1 - \epsilon$, so that $F:S \to R^1$ is of class C^1 on S. By the Whitney Extension Theorem we obtain the result.

<u>Theorem</u> (Whitney [71]). A function $F: Q \to R^1$ is approximately differentiable almost everywhere if and only if to each $\epsilon > 0$ corresponds a closed set K and a function $G:Q \to R^1$ of class C^1 such that $\lambda(Q\setminus K) < \epsilon$ and $F=G$ on K.

The analogues of the other theorems require some auxiliary discussion. To get analogues of BV and

AC we may proceed in several directions all of which are equivalent and have their historical places.

The first approach was given by Tonelli [66] and Cesari [13]. Let $F:Q \to R^1$ be a summable function of n variables $x = (x_1,\ldots,x_n)$. F is said to be of bounded variation, designated BVC, if for each $i = 1,\ldots,n$, there is an equivalent F^i which is of bounded variation in x_i for almost all values of the remaining $n - 1$ variables and this variation function is summable. We take as the F^i those functions in the equivalence classes for which the integrals of the variations are minimized; let the values of these integrals be $\mu_F^1(Q),\ldots, \mu_F^n(Q)$.

By applying the definition to intervals in Q, and extending the consequent set functions, we obtain the variation measures $\mu_F^1(E),\ldots, \mu_F^n(E)$ defined on the Borel sets in Q. Two numerical valued measures obtained from this vector valued measure are of interest. If λ is Lebesgue measure, then α_F is the variation measure corresponding to the vector valued measure $(\lambda, \mu_F^1,\ldots, \mu_F^n)$ and β_F is the variation measure corresponding to the vector valued measure $(\mu_F^1,\ldots, \mu_F^n)$. We shall only use the first one here.

If $F \in$ BVC and the F^i may be chosen to be absolutely continuous in x_i for almost all values of the remaining $n - 1$ variables, then F is in the class analogous to absolute continuity; we designate this class ACT. Tonelli [66] considered the continuous case, and later Goffman [18] discussed the general case.

These classes may also be defined in terms of distributions. The functions in BVC are those whose partial derivatives are measures and the measures are $\mu_F^1, \ldots, \mu_F^n$. The functions in ACT are those for which the measures are absolutely continuous with respect to Lebesgue measure.

Remark 1. It is known [62] that for BVC and ACT a single equivalent function will work for all directions, not just for the coordinate directions. Thus, these notions, are coordinate free.

Remark 2. Since for each $F \in$ BVC the partial derivatives exist almost everywhere, it follows by Stepanov's theorem that the approximate differentials exist almost everywhere. The differentials themselves need not exist anywhere. Indeed, there are functions in ACT which are nowhere continuous (i.e. for every equivalent function). A construction following the lines of the example in Chapter I suffices. Zygmund, his students, followers, and disciples have done much work with the L^p differential which they introduced. Indeed, Calderon and Zygmund [12] have shown that if $F \in$ BVC then its L^p differential exists almost everywhere for some $p > 1$. However, for the application to parametric surface area made in the next chapter it seems that this kind of differential does not suffice but a special sort of approximate differential does.

The third approach involves the notion of area of a nonparametric surface [13], [18]. Start with the set P of piecewise linear functions on Q and let $A(p)$ be the area of the surface given by $p \in P$. If P is metrized by the uniform metric then $A(p)$ is lower semicontinuous on P so that $A(p)$ extends by the Fréchet process [18] to a lower semicontinuous extension $A(f)$ on the completion of P, which is the set of continuous functions on Q. This is the Lebesgue area. If instead of the uniform metric we use the L^1 metric, $A(p)$ still turns out to be lower semicontinuous on P and again using the Fréchet process $A(p)$ extends to a lower semicontinuous "area" functional on the space of equivalence classes of summable functions. Remarkably, it agrees for continuous functions with the Lebesgue area. This is Goffman's form [18] of the area first introduced by Cesari [13]. By applying the definition to intervals in Q and extending the resulting set function to the Borel sets, an area measure is obtained. The functions of finite area are those of type BVC. Those for which the area is given by the formula $A(F) = \int \ldots \int [1 + F_1^2 + \ldots + F_n^2]^{\frac{1}{2}} dx_1 \ldots dx_n$ are of type ACT, and the area measure is α_F.

The equivalence of the three approaches for obtaining BVC and ACT was observed by Krickeberg [30].

It remains for us to discuss the analogue of the intermediate class CBV (continuous functions of bounded variation). The analogous class is taken to be those functions in BVC for which the F^i may be chosen to be continuous in x_i for almost all values of the other variables. This space, which we designate $\mathcal{L}$, was introduced by Goffman [20] as the set of functions of type BVC whose surface area is given by the Hausdorff $n - 1$ dimensional measure of its graph. It is also true that the functions in $\mathcal{L}$ are precisely those functions in BVC which are approximately continuous except on a set of Hausdorff $n - 1$ dimensional measure zero. Since, in R^1, an approximately continuous function of bounded variation is continuous, and sets of Hausdorff 0 dimensional measure zero are empty, this indicates that this class is the natural analogue of CBV [21]. (Thus, approximate continuity is a natural notion here. It is simply an accident that in R^1 continuity appears). There are various other justifications for this notion but they will not be discussed here.

Even though functions in each of the three classes can be everywhere discontinuous, full analogues of the Lusin type approximations for R^1 are valid in R^n for BVC and ACT and weaker analogues hold for $\mathcal{L}$.

For BVC, we have the theorem of Michael [44] in the form given by Goffman [22].

<u>Theorem</u>. A function $F:Q \rightarrow R^1$ is in BVC if and only if for each $\epsilon > 0$ there is a $G \in C^1$ such that $F = G$ except on a set of n dimensional Lebesgue measure less than ϵ and $|\mu_F^i(Q) - \mu_G^i(Q)| < \epsilon$, $i = 1, \ldots, n$.

An equivalent way of stating the conclusion is

$F = G$ except on a set of measure less than ϵ and $|\alpha_F(Q) - \alpha_G(Q)| < \epsilon$.

As an easy consequence we have the analogue of the AC theorem.

<u>Theorem</u>. A function $F \in ACT$ if and only if for each $\epsilon > 0$ there is $G \in C^1$ such that $F = G$ except on a set E with $\alpha_F(E) < \epsilon$ and $\alpha_G(E) < \epsilon$.

The full analogue of the Lusin type approximation theorem for CBV has not been obtained. However, the following partial analogues are known. These analogues are different for both $n = 2$ and for $n \geq 3$. A consequence of these results is the coordinate invariance of $\mathcal{L}$. Thus, functions of type BVC, ACT, and $\mathcal{L}$ are all coordinate invariant.

<u>Theorem</u> [19]. For $n = 2$, if $F \in BVC$ it is in $\mathcal{L}$ if and only if for every $\epsilon > 0$ there is a continuous G such that $F = G$ except on a set E such that $\alpha_F(E) < \epsilon$ and $\alpha_G(E) < \epsilon$.

<u>Theorem</u> [21]. For $n \geq 3$, if $f \in BVC$ it is in $\mathcal{L}$ if and only if for every $\epsilon > 0$ there is an approximately continuous G such that $F = G$ except on a set E such that $\alpha_F(E) < \epsilon$ and $\alpha_G(E) < \epsilon$.

III. The Area Formula: Regular Approximate Differentiation.

The classical work on the area formula for parametric surfaces involves Jacobians and assumes continuous differentiability of the mapping. Because the Jacobians appear in the statement of the formula, it is natural that the general framework consider mappings whose coordinate functions are in Sobolev spaces. Thus, a natural condition for setting up a more general framework for the area formula is that the coordinate functions are in Sobolev spaces. This condition assures the almost everywhere existence of the partial derivatives which are needed for the Jacobians. Observe that this implies, by Stepanov's theorem, that the approximate differential exists almost everywhere. However, even this does not seem to suffice for obtaining the area formula. For the purpose of obtaining the area formula, we introduce a notion of regular approximate differential which does suffice for the area formula, whose existence almost everywhere is then guaranteed when the coordinate functions are in appropriate Sobolev spaces. In this setting the argument for continuously differentiable mappings works when suitably modified.

We use as our model the fact that for a mapping $T: x_i = x_i(u_1, u_2)$, $i = 1,2,3$, of class C^1 from the unit square Q in R^2 into R^3 the area is given by the formula $A(T) = \int_0^1 \int_0^1 J \, du_1 du_2$ where J is the square root of the sum of squares of the three Jacobians involved. Since for partitions of Q of small norm the area of the portion of the surface which spans the image of the boundary of each partition interval is not much less than the area of the linear mapping of the interval given by the differential at its center, it follows that $A(T) \geq \int J$.

The opposite inequality will give us little trouble in our general development. This may seem surprising since in questions involving length and area, the inequality usually goes the other way. However, the Sobolev mappings have absolute continuity properties which allow the necessary limit arguments.

We turn now to an examination of the generality in which the formula holds. Our framework will be that of Sobolev mappings and a Lebesgue type area for such mappings which we define presently.

The Lebesgue area is defined for mappings, just as it is defined for nonparametric surfaces, as the lower semicontinuous extension of the area from the space of piecewise linear mappings with the uniform metric, giving a definition of area for all continuous mappings. Lower semicontinuity no longer holds for the L^1 metric. However, for mappings from R^2 into R^n, $n \geq 2$, area is lower semicontinuous with respect to a metric which yields an extension of Lebesgue area to all mappings which are continuous, as functions of each variable, for almost all values of the other variable. Such mappings are called linearly continuous. For mappings of R^m, $m>2$, into R^n, $n \geq m$, even this fails. However, an area is defined in a similar way for all mappings which are continuous as functions of $n - 1$ variables for almost all values of the other variable. Such mappings are called $m - 1$ continuous [23].

We now define Sobolev mappings. Let Q be the unit cube in R^m. A mapping $T: x_i = x_i(u_1, \ldots, u_m)$, $i = 1, \ldots, n$ of Q into R^n is a Sobolev mapping if each x_i is in some $W_1^{p_i}$ and the sum of the reciprocals of any m of the p_i is ≤ 1. It follows that every Sobolev mapping is linearly continuous and that for any Sobolev mapping $\int J \, du_1 \ldots du_m < \infty$, where J is the square root of the sum of the squares of the Jacobians of the mappings induced by T into the m dimensional coordinate subspaces.

For $m = 2$, the above Lebesgue type area is defined for every Sobolev mapping and we indicate why in this case we always have $A(T) = \int J \, du_1 du_2$. In the first place, for any m, if $A(T)$ is defined, it may be shown using regularizers that $A(T) \leq \int J \, du_1 \ldots du_m$.

For the opposite inequality ($m = 2$) we need a companion to the Stepanov theorem, which holds for functions of two variables. This theorem was obtained by Rado and by Sforza-Dragoni [14]. The theorem says that if $F: R^2 \to R^1$ has partial derivatives almost everywhere then the approximate partial derivative exists in a stronger sense. In fact, the set of density one with respect to which it exists at x is composed of the boundaries of oriented squares centered at x. That this should be so is quite natural. Such an approximate differential is called a regular approximate differential.

<u>Theorem</u>. If $F: R^2 \to R^1$ has partial derivatives almost everywhere then it has a regular approximate differential

almost everywhere.

We indicate how this fact yields the area formula for Sobolev mappings of $Q \subset R^2$ into R^n, $n \geq 2$. Let T be such a mapping. We may follow the idea of the argument for C^1 mappings noted above. For each $\epsilon > 0$, we can find, using the regular approximate differentiability almost everywhere, applying Lusin's theorem to this differential, and applying the Vitali covering theorem, a finite set of pairwise disjoint squares of area sum greater than $1 - \epsilon$ so that the sum of the areas of T over these squares exceeds the sum of the formula integrals over these squares minus ϵ. It follows that $A(T) \geq \int J \, du_1 du_2$. Thus, we have:

<u>Theorem</u> [23]. If T is any Sobolev mapping from the unit square Q into R^n, $n \geq 2$, then

$$A(T) = \int J \, du_1 du_2.$$

The case $m > 2$ is more complicated in some respects. We first note that if $F \in W_1^p$, $p > m$, is a real function on $Q \subset R^m$, then F is differentiable almost everywhere. This was proved for $m = 2$ by Cesari and for all m by Calderon and by Serrin [62]. Moreover, by the Sobolev imbedding theorem, such an F is continuous and actually satisfies a Hölder condition.

Sobolev mappings are linearly continuous, but for $m > 2$ they need not be $m - 1$ continuous so that the area is not even defined. However, if for a Sobolev mapping $T: x_i = x_i(u_1,\ldots,u_m)$, $i = 1,\ldots, n$ we have $p_i > m - 1$, $i = 1,\ldots,n$ then it follows that T is both $m - 1$ continuous and has a regular approximate differential. (A mapping from $Q \subset R^m$ into R^n, $n \geq m$, is said to have a regular approximate differential at u if a set of density one at u with respect to which the differential exists consists of the boundaries of oriented m cubes with u as center). The existence almost everywhere of the regular approximate differential then follows by an application of the Calderon-Serrin Theorem to $m - 1$ dimensional hyperplanes.

The two dimensional argument may then be applied to the higher dimensional case to give the following result:

<u>Theorem</u> [25]. If $T: x_i = x_i(u_1,\ldots,u_m)$, $i = 1,\ldots, n$, is a mapping of $Q \subset R^m$ into R^n, $n \geq m$, such that each $x_i \in W_1^{p_i}$, with $p_i > m - 1$ and $\sum_{j=1}^{m} p_{i_j}^{-1} \leq 1$ whenever $1 \leq i_1 < i_2 < \ldots < i_m \leq m$, then

$$A(T) = \int J \, du_1 \ldots du_m .$$

IV. Approximate differentiation in R_1.

In the previous sections we discussed some of the ways in which approximate differentiation arises naturally. We focused primarily on functions or mappings involving Euclidean spaces of two or more dimensions, and we were concerned with the theory of "almost-everywhere" approximate differentiation. For functions of one real variable, the almost-everywhere theory has been well developed for quite some time. In fact, much of it can be found in Sak's book [60]. One finds there an exposition of the role of approximate differentiation in the theory of Denjoy-Khintchine integration and of the ways in which approximate differentiation relates to questions of growth of a function (e.g., questions concerning monotonicity).

More recently, particularly during the last 15 years or so, a number of authors have focused on questions concerning the "everywhere" theory of approximate differentiation of functions of one real variable. Here much of the emphasis is on the structure or behavior of approximate derivatives (as related to the structure or behavior of ordinary derivatives) rather than on applications to other parts of mathematics. What triggered this emphasis is clear. In 1950, Zahorski [73] provided a penetrating study of the delicate structure of ordinary derivatives. He was concerned primarily with the longstanding problem of characterizing derivatives. While he was not able to obtain a complete characterization, he was able to find certain necessary conditions and certain sufficient conditions for a function to be a derivative and he signaled certain important classes of functions related to the class of derivatives. This work of Zahorski's was the starting point of a number of works dealing with the behavior of derivatives. Then, in 1960, Goffman and Neugebauer [24] provided a unified treatment of much of what was known at the time about the behavior of approximate derivatives. These known results suggested that approximate derivatives behave very much like ordinary derivatives. It was thus natural to ask whether those properties of derivatives which came out of Zahorski's work were also shared by approximate derivatives. And this led a number of authors to study the behavior of approximate derivatives in detail. We shall see that virtually all properties known to be shared by all derivatives are also shared by all approximate derivatives. Yet, in some sense, an approximate derivative which is not itself a derivative, must be <u>much</u> more wildly behaved than any derivative. It would certainly be of interest to find a property of derivatives which is not also a property of approximate derivatives. Because we believe this to be an important problem, we believe it is desirable to clarify the problem a bit. In a sense there are, of course, properties that distinguish derivatives from approximate derivatives. The definitions themselves offer such a distinction, as do the classes of primitives - for example, a differentiable function must be continuous whereas an approximately differentiable function need only be approximately continuous. Furthermore, a derivative is always integrable in the Denjoy-Perron sense; an approximate derivative need not be. Thus, our probelm is not entirely well defined. We hope our development of approximate derivatives will serve to clarify the problem.

In this chapter we shall discuss the recent work related to the one-variable theory of approximate differentiation. For the earlier work, we refer the reader to Saks [60]. We shall discuss the structure and behavior of approximate derivatives, compare approximate differentiation with ordinary differentiation, discuss (briefly) certain generalizations, and state a few open problems. To fix ideas and notation, we begin with some fundamental concepts.

Let λ denote Lebesgue measure on the line. A point x_o is called a point of density of a set E if

$$\lim_{\substack{h \to 0^+ \\ k \to 0^+ \\ h+k \neq 0}} \frac{\lambda(E \cap [x-h, x+k])}{h+k} = 1 .$$

According to Lebesgue's Density Theorem, almost every point of a measurable set is a point of density of that set.

A function F is said to be approximately differentiable at a point x with approximate derivative $F'_{ap}(x)$ at x if there exists a set E having x as a point of density such that $F'_{ap}(x) = \lim_{\substack{y \to x \\ y \in E}} \frac{F(y) - F(x)}{y - x}$, and this limit is finite. A function f defined on an interval I is called a derivative (an approximate derivative) if there exists a differentiable (approximately differentiable) function F such that $F'(x) = f(x)$, $(F'_{ap}(x) = f(x))$ for all $x \in I$. We denote the class of all derivatives on I by Δ' and the class of all approximate derivatives on I by Δ'_{ap}.

1. Basic behavior of approximate derivatives.

Among the properties of ordinary derivatives, two are well known and easy to prove: each derivative is in the first class of Baire (i.e., is the pointwise limit of a sequence of continuous functions), and has the Darboux property (sometimes called the Intermediate Value Property). The first result follows immediately from the observation that if $F' = f$, then

$$f(x) = \lim_{n \to \infty} \frac{F(x + \frac{1}{n}) - F(x)}{\frac{1}{n}} \quad \text{for all } x$$

(a slight modification being necessary if f is defined on some bounded interval). To verify the second assertion we can easily reduce the problem to the case $F'(a)<0<F'(b)$ from which it follows that F achieves a minimum value at some point $x_0 \in (a,b)$. At this point $F'(x_0) = 0$.

In 1927 Khintchine [29] proved that each approximate derivative has the Darboux property and in 1938 Tolstoff [64] proved that each approximate derivative is in the first class of Baire. Neither of these proofs was simple. Using ideas involving convergent interval functions, Goffman and Neugebauer [24] were able to obtain relatively simple proofs of these theorems. Their techniques also were useful in proving another result of Tolstoff's [64] according to which a function F which is approximately differentiable on an interval is actually differentiable except on some nowhere dense set. A direct development of this last result, along with others, can follow these lines. One first shows that for a monotonic function, the extreme derivates (Dini derivatives) coincide with the extreme approximate derivates (Misik [46]). Thus, a monotonic function is differentiable at each point at which it is approximately differentiable. One can then infer without difficulty that an approximate derivative which is bounded above or below on an interval I is an ordinary derivative on I. Since each approximate derivative f is in the first class of Baire, it is continuous on some dense set D. If $x \in D$, then f is bounded in some neighborhood N of x and is therefore an ordinary derivative on N. Tolstoff's result follows. (Our discussion also shows that an approximate derivative f cannot be dominated by a derivative g without itself being a derivative, because the difference $h = f - g$ is an approximate derivative which is bounded above or below.)

Since a monotonic function F which is approximately differentiable at a point x_0 must be differentiable at x_0, one might conjecture that the same is true for a function of bounded variation. This is not the case, however. It is easy to construct a function F meeting the following conditions on [0,1].

(i) F is continuous.

(ii) $F(x) = x$ if $x = \frac{1}{2^n}$, $n = 1,2,3,\ldots$

(iii) $0 \leq F(x) \leq x$ for all $x \in [0,1]$

(iv) $F(x) = 0$ for $x \cup I_k$, where $\{I_k\}$ is a sequence of pairwise disjoint intervals converging to the origin and having the origin as a point of density.

(v) The graph of F has the shape of a spike on each interval lying between two successive intervals I_k.

Then the total variation of F equals 2, $F'_{ap}(0) = 0$ but F is not differentiable at the origin. (By rounding off corners in (v) we can even make F differentiable in all of $(0,1]$].

We mentioned in Chapter I that the approximate partial derivates reflected the measurability properties of their primitives. Some results of a precise nature have recently been obtained for the one variable case by Misik [47].

2. <u>The set of points of differentiability of an approximately differentiable function</u>.

According to Tolstoff's result, an approximately differentiable function F must be differentiable at each point of some dense, open set. Let S be the set $\{x\colon F$ is not differentiable in any neighborhood of $x\}$. It is clear that S is closed and nowhere dense. We shall call this set the <u>singular set</u> for F. Let $D = \{x\colon F$ is differentiable at $x\}$. It is clear that $D \supset \sim S$. We shall be concerned with the behavior of $F'_{ap} = f$ on D and on ~S and we shall see that while the behavior is in some sense "good" on D and on ~S, any "wildness" of f is already apparent on D and on ~S.

We begin with the following question, which at this point we word vaguely: how wild can the behavior of f be on the singular set S in comparison with its behavior on ~S? To put the question into perspective we recall that f has the Darboux property and is in the first class of Baire. This, by itself, puts little restriction on f. It is possible for such a function to vanish on the complement of a nowhere dense set without vanishing identically [11], thus exhibiting wild behavior on the nowhere dense set and calm behavior on the complement. It is also easy to construct examples where the reverse is true.

Our vaguely-stated question was answered in a series of papers [68], [52], [54]. In 1969 Weil [68] showed that an approximate derivative f maps D onto a dense subset of the range of f. O'Malley [52] extended this result to show that the set f(D) is actually the full range of f. (In particular, therefore, f possesses the Darboux property on D.) Finally, O'Malley and Weil [54] showed that more is true. If f takes on the values M and -M on some interval I, then there is a single component interval of ~S on which f takes on those values.

It follows from this result that an approximately differentiable function is determined by its values on a dense set. To see this, let F and G be approximately differentiable on an interval I and suppose $F = G$ on some dense set E. Let $H = F - G$. Then H is approximately differentiable on I and $H = 0$ on E. Let S be the singular set for H. If $S = \emptyset$, H is differentiable and therefore vanishes on I. If $S \neq \emptyset$, then H'_{ap} is unbounded above and below on I.

It follows from the theorem of O'Malley and Weil that there exists a point x_0 in $\sim S$ such that $H'_{ap}(x_0) = H'(x_0) \neq 0$, from which we infer that H cannot vanish on the dense set E.

Another immediate corollary (to which we shall refer in Section 4) is that if F is approximately differentiable on I and G is differentiable on I with $F' = G'$ off the singular set for F, then F is differentiable and $F' = G'$ on I.

We have seen that an approximate derivative f which is not an ordinary derivative must oscillate wildly (it must be unbounded above and below in each neighborhood of each point of its singular set). This wild oscillatory behavior cannot be achieved on S alone; it must be achieved on $\sim S$ whether or not it is achieved on S. This may lead one to suspect that questions concerning the summability of f will be determined by the behavior of f off S. This is not entirely true. Fleissner and O'Malley [16] have recently shown that there exist approximate derivatives on [0,1] which are summable on $\sim S$ but not summable on all of [0,1].

On the other hand, they also proved that f will be summable on [0,1] if and only if f is summable on D. Thus, while $\sim S$ dominates the oscillatory behavior of f, it takes a larger set to determine whether or not f is summable.

3. <u>Associated sets</u>.

Let f be a real valued function of a real variable. For real numbers α and β let

$$E_\alpha = \{x: \ f(x) > \alpha\}, \ E^\beta = \{x: \ f(x) < \beta\}$$

and let

$$E_\alpha^\beta = E_\alpha \cap E^\beta .$$

Sets of the type E_α and E^β are called associated sets for the function f.

Many classes $\mathcal{A}$ of functions can be characterized in terms of associated sets. This means that there exists a family S of subsets of the line such that $F \in \mathcal{A}$ if and only if all the associated sets of f are members of S. The chart below exhibits some of the better known characterizations of classes of functions in terms of associated sets.

$\mathcal{A}$	S
continuous	open
Baire	Borel
measurable	measurable
Baire 1	F_σ

A major purpose of Zahorski's work [73] was to obtain a characterization of Δ' in terms of associated sets. Towards this end he defined a hierarchy of classes of functions $\mathcal{M}_0 \supset \mathcal{M}_1 \supset \dots \supset \mathcal{M}_5$ each defined in terms of associated sets. As the class became smaller, the associated sets become "fatter" near each of their members. For example, a function is in $\mathcal{M}_0$ provided each associated set E is of type F_σ and for $x_0 \in E$ and I any open interval having x_0 as an endpoint, the set $I \cap E$ is nonempty. For membership in $\mathcal{M}_1$, the set $I \cap E$ has to be nondenumerable, and for membership in $\mathcal{M}_2$ this set must have positive measure. The definitions of $\mathcal{M}_3$ and $\mathcal{M}_4$ are rather complicated to state and we shall omit the statements. For $\mathcal{M}_5$, we require x_0 to be a point of density of E. It turns out that $\mathcal{M}_0 = \mathcal{M}_1 = \mathcal{DB}_1$, the class of Darboux functions in the first class of Baire and that $\mathcal{M}_5$ consists of the class of approximately continuous functions. "Full fatness" of course, is achieved only by continuous functions.

Since $\Delta' \subset \mathcal{DB}_1 = \mathcal{M}_0$ and since each bounded approximately continuous function is a derivative, it was natural to attempt to find exactly how the class of derivatives (or bounded derivatives) fits into the scheme. Zahorski showed that $\Delta' \subset \mathcal{M}_3$ and that each bounded derivative is in $\mathcal{M}_4$. But he was unable to characterize Δ' or the class of bounded derivatives in terms of associated sets. (We now know [7] that such a characterization is impossible.)

Prior to Zahorski's work, it was known that if $f \in \Delta'$ then each set E_α^β is either empty or has positive measure. This is not quite the same thing as saying $\Delta' \subset \mathcal{M}_2$, but it is close, and the earlier proofs could be modified to give that result.

Where do approximate derivatives fit into the scheme? Marcus [41] showed that if $f \in \Delta'_{ap}$, then E_α^β is empty or of positive measure for each $\alpha < \beta$. Once again, the inclusion $\Delta'_{ap} \subset \mathcal{M}_2$ is readily obtained. Then, in 1965, Weil [67] showed that $\Delta'_{ap} \subset \mathcal{M}_3$. Weil also showed in [69] that each approximate derivative possessed a property of derivatives slightly stronger than $\mathcal{M}_3$. These results suggest the problem of finding a class $\mathcal{M}$ of functions defined in terms of associated sets such that $\Delta' \subset \mathcal{M}$ but $\Delta'_{ap} \not\subset \mathcal{M}$.

Such a family does not exist! In fact, Preiss [57] defined a class which he called $\mathcal{M}_3^*$. This class is contained in $\mathcal{M}_3$. (We shall not give the very complicated definition.) Preiss showed that $\Delta'_{ap} \subset \mathcal{M}_3^*$ and that given any associated set E for functions in $\mathcal{M}_3^*$, there exists an $f \in \Delta'$ such that $E = \{x: f(x) > 0\}$. Thus E is an associated set for some derivative. It follows that the family of associated sets for $\mathcal{M}_3^*$ is simultaneously the family of associated sets for Δ' and for Δ'_{ap} .

Thus, we cannot use associated sets (at least not by themselves) to pick out the class of ordinary derivatives from the class of approximate derivatives. This adds further strength to the statement that approximate derivatives behave very much like derivatives.

4. Representations, decompositions and extensions.

We have seen that approximate derivatives possess all the known properties of derivatives. On the other hand, an approximate derivative cannot be dominated by a derivative without it being a derivative itself. In fact, if $F'_{ap} = f \in \Delta'_{ap} \sim \Delta'$, then f cannot be dominated by a derivative on any interval containing points of the singular set S. Taken together, these results suggest that it might be possible to represent approximate derivatives by countably many derivatives in some manner or other. Perhaps Tolstoff's result can be useful here. Since $F'_{ap} = F'$ on each interval contiguous to S, it might be possible to "paste together" countably many derivatives in such a way as to make up f. Or perhaps some other techniques can be used.

The first result in this direction was obtained by O'Malley [53]. Let $F'_{ap} = f$ on [0,1]. (To avoid trivial cases, assume $f \in \Delta'_{ap} \sim \Delta'$.) Then there exists a sequence of perfect sets $\{H_n\}$ and a sequence of differentiable functions $\{G_n\}$ such that $G_n = F$ on H_n, $G_n' = f$ on H_n, and $[0,1] = \cup H_n$. The sets H_n cannot be disjoint, of course, because an interval cannot be expressed as a union of countably many pairwise disjoint closed sets. Suppose, however, that we write

$A_1 = H_1$ and $A_n = H_n \sim \bigcup_{k < n} H_k$ if $n > 1$. Then the

sets A_n are pairwise disjoint and $[0,1] = \cup A_n$. Also, $G_n = F$ and $G_n' = f$ on A_n. We have thus decomposed F into countably many restrictions of differentiable functions and f into countably many restrictions of derivatives. The restrictions are, of course, to the sets A_n, each of which is the difference of two closed sets. Is it possible for one of the sets A_n to contain an interval I contiguous to the singular set S? The answer, is, unfortunately, "no". This follows directly from our remark in Section 2 applied to the closure of I: from $F' = G'$ on I, we would have to infer F differentiable on $c\ell\, I$, which is impossible because $c\ell\, I \cap S \neq \emptyset$.

This last observation also can be interpreted in terms of f rather than F. Even though an approximate derivative is an ordinary derivative on each interval I contiguous to its singular set, it cannot be extended from I to a derivative on $c\ell\, I$. It is, therefore, perhaps curious that it is always possible to extend an approximate derivative from its <u>singular</u> set. In precise terms, if $f \in \Delta'_{ap}$ with singular set S, there exists a function $g \in \Delta'$ such that $g = f$ on S. Actually, f can be extended to a derivative from any nowhere dense set [1].

Another method of representing approximate derivatives by derivatives was obtained in [1]. Here it was shown that to each $f \in \Delta'_{ap}$ corresponds three differentiable functions g, h, and k such that $f(x) = g'(x) + h(x)k'(x)$ for all $x \in I$. Thus, in particular, each approximate derivative is in the algebra generated by the class of derivatives.

One other representation for approximate derivatives is worth mentioning. In [50], O'Malley developed a very general notion of derivative which he called the <u>selective derivative</u>. One first selects one point x_I from each subinterval I of $[0,1]$. The resulting set of points is called a selection s. If $\lim \dfrac{F(x_{[x_0,x_0+h]})-F(x_0)}{x_{[x_0,x_0-h]}-x_0}$ exists, we denote that limit by $sF'(x_0)$ and call it the selective derivative (relative to the selection s) at the point x_0. If F has a selective derivative for all x in $[0,1]$, we say F is selectively differentiable and call sF' the selective derivative of F (relative to s).

Thus, each selection s determines a class of selectively differentiable functions and a class of selective derivatives $\Delta'_s \supset \Delta'$.

O'Malley showed, among other things, that each approximate derivative is a selective derivative relative to some selection, thus approximate derivatives can be realized as selective derivatives. In the other direction, no matter which selection s is chosen, each s-differentiable function F is approximately differentiable a.e. and $sF' = F'_{ap}$ a.e. (O'Malley also showed that selective derivatives possess properties shared by various other generalized derivatives. Properly exploited, his ideas may shed a good deal of light on why most generalized derivatives behave as they do. Since a development would take us too far afield from our main purposes, we shall not attempt an exposition here.)

5. Algebraic structure of the classes Δ' and Δ'_{ap}.

We have emphasized the fact that the individual functions in Δ'_{ap} possess all known properties shared by individual functions in Δ'. What about the entire classes Δ' and Δ'_{ap}? Do they possess similar algebraic and topological structures?

It is easy to verify that both Δ' and Δ'_{ap} are closed under sums. In 1921 Wilkosz [72] proved that the square of a derivative needs not be a derivative. Thus Δ' is not closed under multiplication, nor under outside composition with continuous functions. The same is true of functions in Δ'_{ap}. In fact, if f and f^2 are both in Δ'_{ap}, then they are both in Δ' [8]. Both Δ' and Δ'_{ap} are closed under uniform limits. For Δ', this is a standard result of elementary real analysis. To verify the statement for Δ'_{ap}, let $\{f_n\}$ be a sequence of functions in Δ'_{ap} converging uniformly to a function f. For N sufficiently large, the functions $g_n = f_N - f_n$ $(n \geq N)$ are bounded and in Δ'_{ap}. Therefore $g_n \in \Delta'$ for $n \geq N$. It follows that

$g \equiv \lim_{n \to \infty} g_n = f_N - f$ is a uniform limit of derivatives and is therefore itself a derivative. Thus $f = f_N - g$ is an approximate derivative.

Regarding pointwise limits, the situation is this: Since $\Delta'_{ap} \subset \mathcal{B}_1$ (the functions in the first class of Baire), it is clear that the pointwise limit of a sequence of functions in Δ'_{ap} is in $\mathcal{B}_2$, the second class of Baire. Preiss [55] has shown that each $f \in \mathcal{B}_2$ is the pointwise limit of a sequence of functions in Δ'. Since $\Delta' \subset \Delta'_{ap}$, the same is true *a fortiori* for Δ'_{ap}.

What can one say about the effects of homeomorphic changes of scale on the classes Δ' and Δ'_{ap} ? More precisely, how can one characterize the classes of the form $f \circ h$ where $f \in \Delta'$ (or Δ'_{ap}) and h is a homeomorphism of the domain of f with itself? We can ask the same questions for the classes $h \circ f$, where h is now a homeomorphism of R with itself. The first question was answered by Maximoff [43]. He proved that every function in $\mathcal{DB}_1$ (the Darboux functions in the first class of Baire) can be represented in the form $f \circ h$, $f \in \Delta'$. It follows, a fortiori, that the same is true for Δ'_{ap}. Maximoff actually proved this result in several different papers, but all of his proofs are quite complicated and not entirely clear. A new and more transparent proof has recently been advanced by Preiss [58]. (Choquet [15] proved the related result that each semicontinuous function with the Darboux property is of the form $f \circ h$, $f \in \Delta'$.) If one suitably restricts the allowable homeomorphisms, then Δ' is closed under inner compositions with homeomorphisms. Some recent results related to this question can be found in [8], [26], and [34].

Regarding compositions with outer homeomorphisms, much less is known. Some work has been done in recent years concerning compositions of the form $h \circ f$, $f \in \Delta'$: to each non-linear h corresponds an $f \in \Delta'$ such that $h \circ f \notin \Delta'$; if $f \in \Delta'$, f bounded, then $h \circ f \in \Delta'$ for h

strictly convex if and only if f is approximately continuous [9]; there exists $f \in \Delta'$ such that $h \circ f \notin \Delta'$ for every h which is nonlinear on every interval [9]. Furthermore, it is clear that $h \circ f \in \mathfrak{M}_3^*$ for each $f \in \Delta'$ because $\mathfrak{M}_3^*$ is defined in terms of associated sets, and the family of associated sets is invariant under outside composition with a homeomorphism. To the best of our knowledge, this last result is the only known result of this type for Δ'_{ap}. Work concerning compositions of the form $h \circ f$, $f \in \Delta'_{ap}$ seems to have not yet been done. It would be of interest to study those functions which can be expressed in the form $h \circ f$, $f \in \Delta'_{ap}$ (or $f \in \Delta'$). At the moment we know, because of Preiss' work, that each of the resulting classes is in $\mathfrak{M}_3^*$. How does one characterize these classes? One can also ask for restrictions on the homeomorphisms (e.g., Lipschitz conditions, differentiability, etc.) such that $h \circ f \in \Delta'$ for all $f \in \Delta'$ or $h \circ f \in \Delta'_{ap}$ for all $f \in \Delta'_{ap}$.

We know of no work in these directions. Iosifescu [26] has shown that if $f \in \Delta'$ then $f^2 \in \Delta'$ if and only if each point x_0 in the domain of f is a Lebesgue point of the second kind for f: i.e.,

$$\lim_{h \to 0} \frac{1}{n} \int_{x_0}^{x_0+h} |f(x) - f(x_0)|^2 dx = 0 \quad \text{for all } x_0.$$

Although the squaring function is not a homeomorphism, this result may give a clue as to how to proceed.

We can ask the analogous question for differentiable or approximately differentiable functions (instead of derivatives or approximate derivatives). A development of recent work on such questions (as well as more detail on the questions we just discussed) can be found in [10] and [11].

We close this section by mentioning, for purposes of comparison, that the class $\mathcal{DB}_1$ is closed under uniform limits, composition with continuous functions, (both inside and outside compositions), but not under sums or products.

6. Infinite approximate derivatives.

To this point we have restricted our attention to finite approximate derivatives. The situation for approximate derivatives which may take on infinite values is somewhat more complicated, although some of the structure remains. Because of the complexity of the subject, we shall merely give an indication of the present state of knowledge.

Let F be a function for which $F'_{ap}(x)$ exists (finite or infinite) for each point x in some interval I. We ask: what does this imply about the functions F

and $f = F'_{ap}$? Recent work shows that F possesses at least some desirable structure, and that the extent to which f possesses the properties we discussed for finite approximate derivatives depends to some degree on what one assumes about F.

A natural assumption to place on F when dealing with approximate derivatives might be that of approximate continuity. (If F'_{ap} is finite, then F is approximately continuous.) But it is easy to construct F so that $F'(x_0) = \infty$, $F'(x) = 0$ for $x \neq x_0$ and F has a "jump" at x_0. Then F does not even have the Darboux property (neither does F'). This function F is in the first class of Baire, as is $F'_{ap} = F'$. Does this always happen? Preiss [56] provides an example of a function F such that F'_{ap} exists on [0,1] but F is not in the first class of Baire. On the other hand, he proves that F'_{ap} must be in the first class of Baire, even if F is not. (This is true even if F does not have the Darboux property.) The Preiss example F is in the second class of Baire. Must this always happen? Zahorski showed it must.

What about the Darboux property? Misik [45] used Preiss' result (that $F'_{ap} \in \mathcal{B}_1$) to show that if F has the Darboux property, so does F'_{ap}. (Preiss actually proved a slightly stronger result: he assumed a bit less than the Darboux property for F and concluded that F'_{ap} possesses the Denjoy property $\{x:\alpha < F'_{ap} < \beta\}$ is either empty or has positive measure. It follows readily that $F'_{ap} \in \mathcal{M}_2$).

To summarize, F must be in $\mathcal{B}_2$, but not necessarily in $\mathcal{B}_1$, and might fail to have the Darboux property. On the other hand, F'_{ap} is always in $\mathcal{B}_1$ and, whenever F has the Darboux property, so does F'_{ap}. In that case, $F'_{ap} \in \mathcal{M}_2$.

Actually more is known. In his deep study of associated sets for derivatives [57] Preiss did not restrict himself to finite derivatives. Among other things, he defined a family M^* of subsets of the line and then let $M_2^* = M^* \cap M_2$ and $M_3^* = M^* \cap M_3$. (Here M_2 and M_3 are the families of sets studies by Zahorski [73].)

Preiss proved that M^* is the family of associated sets for the class of (possibly infinite) approximate derivatives, as well as the family of associated sets for the class of (possibly infinite) derivatives; that M_2^* is the family of associated sets for the class of (possibly infinite) approximate derivatives of Darboux functions, as well as the family of associated sets for the class of (possibly infinite) derivatives of increasing absolutely continuous functions; and that M_3^* is the family of associated sets for the class of finite approximate derivatives as well as for the class of finite derivatives of increasing absolutely continuous functions.

Thus, Preiss' work together with Zahorski's, completely characterizes the families of associated sets for certain subclasses of Δ' and Δ'_{ap}. Observe that some of the results we mentioned earlier in this section are consequences of these characterizations. Because the definition of the family M^* is extremely complicated, we shall not state it here.

Certain other recent works related to the material of this section may be of interest to the reader. We mention, in particular, the papers of Krzyzewski [31], Kulbacka [32], Lipinski [38] and Matysiak [42].

7. Approximate differentiability structure of typical continuous functions.

Let $\mathcal{C}$ denote the space of continuous functions on [0,1] furnished with the sup norm, $\|f\| = \max\{|F(x)|: 0 \le x \le 1\}$. It has been known for some time that the typical function in $\mathcal{C}$ exhibits pathological behavior with respect to (ordinary) differentiation properties. For example, the typical continuous function is nowhere differentiable. We use the term "typical" here to mean that the set of functions which do not have the desired property (e.g., of being nowhere differentiable) forms a first category subset of the complete metric space $\mathcal{C}$.

What can we say about the behavior of the typical continuous function with respect to approximate differentiation properties?

In 1934, Jarnik [28] proved that the typical continuous function F is nowhere approximately differentiable. Around the same time [27], he clarified the behavior of such an F a bit by showing that every extended real number is an essential derived number a.e. Stated precisely, this means that there exists a set Z of Lebesgue measure 0 such that if $x \in \sim Z$ and $-\infty \leq \gamma \leq \infty$, then there exists a set $E(x)$ having upper density 1 at 0 such that

$$\lim_{\substack{h \to 0 \\ h \in E(x)}} \frac{F(x+h)-F(x)}{h} = \gamma.$$

Suppose, now, that f is an arbitrary function defined on [0,1]. It follows immediately from Jarnik's result that for a typical continuous function F, there corresponds to almost every $x \in [0,1]$ a set $E(x)$ having unit upper density at the origin such that

$$f(x) = \lim_{\substack{h \to 0 \\ h \in E(x)}} \frac{F(x+h)-F(x)}{h}.$$

Thus, one might say that a typical F is a "universal anti-approximate derivative a.e.". Recently, Scholz [61] proved that if we consider only measurable functions

f, then the sets $E(x)$ can all be chosen to be the same: that is, given f, there exists a set E having unit upper density at the origin, such that

$$(*) \qquad f(x) = \lim_{\substack{h \to 0 \\ h \in E}} \frac{F(x+h)-F(x)}{h} \quad \text{for almost all } x \in E .$$

In addition to improving Jarnik's theorem, this remarkable result also extends a theorem of Marcinkiewicz [40] according to which (*) holds when E is replaced by some appropriately chosen sequence $\{h_n\}$ converging to 0.

8. Monotonicity.

Suppose we wished to show that some function F is nondecreasing on an interval I. A first approach might be to invoke the theorem of elementary calculus to the effect that if $F' \geq 0$ on I, then F is nondecreasing. But what if we cannot invoke this theorem? Perhaps F is not differentiable, (or it is, but we cannot establish this). Or, perhaps we cannot establish that $F' \geq 0$ on I. We would like a theorem which guarantees monotonicity under weaker hypotheses than those of the theorem of elementary calculus. In 1928 and 1930, Goldowski and Tonelli (see Saks [60]) provided such a theorem:

Theorem. Let F satisfy the following conditions on an interval I:

(i) F is continuous

(ii) F' exists (finite or infinite) except perhaps on a denumerable set.

(iii) $F' \geq 0$ a.e.

The F is nondecreasing on I.

(The negative of the Cantor function shows that we cannot drop condition (ii) in the statement of the theorem.)

In recent years there have been literally dozens of theorems similar in spirit to the Goldowski-Tonelli theorem. In each case, one assumes that F meets some regularity condition (e.g., continuity, approximate continuity, or membership in $\mathcal{B}_1$), that some generalized derivative exists except perhaps in a set A which is small in some sense, and that this generalized derivative is nonnegative except perhaps in a set B which is also small in some sense. (The senses in which A and B are assumed small may differ.) One then concludes that F is nondecreasing. A reasonably complete development of this subject would take us too far afield (but see [11] for such a development). We shall restrict our attention to the case in which the generalized derivative is the approximate derivative.

The first direct improvement of the Goldowski-Tonnelli theorem in the setting of approximate differentiation was advanced by Tolstoff in 1939 [65]. He assumed only that F is approximately continuous and that conditions (ii) and (iii) are met by F'_{ap}, and concluded that F is continuous and nondecreasing. (Because of Misik's theorem (Section 1) it follows that conditions (ii) and (iii) are also met by F'.) More recently ([5], [6], [63]) it was established that condition (i) can be replaced by the still weaker condition that F be a Darboux function in the first class of Baire, with monotonicity still following.

These theorems can be viewed as ones in which the approximate derivative substitutes for the derivative, but they do not give an indication of the full extent to which such substitutions are possible. Recently, O'Malley and Weil [54] provided a theorem which is a step in that direction. Basically they proved that if a condition is sufficiently strong to guarantee monotonicity for each differentiable function meeting the condition, then it is also sufficiently strong to guarantee monotonicity for each <u>approximately differentiable</u> function meeting the condition. This theorem is very general because it does not specify the nature of the condition. The condition need not be one which is stated in terms of the behavior of the derivative. If it is, however, it allows us to replace it with the corresponding statement for the approximate derivative and still conclude that the function is monotonic. For example, from the fact that a differentiable function whose derivative is nonnegative a.e. must be nondecreasing, follows the fact that an approximately differentiable function whose approximate derivative is nonnegative a.e. must be nondecreasing. To see this one need only observe that for a differentiable function, the approximate derivative is the derivative, and then apply the O'Malley-Weil theorem.

The O'Malley-Weil theorem requires differentiability or approximate differentiability everywhere. It would be worthwhile to formulate a more general theorem of their type in which this requirement is relaxed.

Related to questions of monotonicity are questions of constancy. These can often be formulated in terms of the notion of a stationary set for a class of functions. Let $\mathcal{H}$ be a class of functions. A set E is called a stationary set for $\mathcal{H}$ if each function in $\mathcal{H}$ which vanishes on E must vanish identically. (Thus, the stationary sets for the class of continuous functions are the dense sets, while each singleton set is a stationary set for the class of constant functions.) The family of stationary sets for a class $\mathcal{H}$ provides some sort of measure of the size of $\mathcal{H}$: If $\mathcal{H}_1 \subset \mathcal{H}_2$, then each stationary set for $\mathcal{H}_2$ is also a stationary set for $\mathcal{H}_1$. Since $\Delta' \subset \Delta'_{ap}$, and there are many functions in $\Delta'_{ap} \sim \Delta'$, one might expect Δ' to possess some stationary sets which Δ'_{ap} does not possess. This does not turn out to be the case. Boboc and Marcus [4] proved that the stationary sets for Δ are the sets whose complement have inner measure zero. These sets also comprise the stationary sets for the class of approximate derivatives (possibly infinite) of Darboux functions. (This result is due to Preiss [56].) This last class contains Δ'_{ap} and it follows in particular that Δ' and Δ'_{ap} have the same family of stationary sets.

Once again, we have been unable to find a way of distinguishing Δ' from Δ'_{ap}.

9. Additional remarks.

We discuss briefly a few additional topics related to approximate differentiation.

The approximate derivative is, of course, a generalization of the ordinary derivative. There are many other such generalized derivatives, some of which can also be viewed as generalizations of the approximate derivative. One such generalized derivative is the selective derivative which we discussed in Section 4. Another is Denjoy's preponderant derivative. Here, instead of requiring the difference quotient to approach a limit through a set of density one at x, we require this to happen through a set exhibiting only a "preponderance of density" at x. The term "preponderance of density" can be interpreted in several ways which are not quite equivalent. Whichever interpretation one takes, the resulting derivative is more general than the approximate derivative. This notion of preponderant derivative has found only limited use.

Other generalizations can be obtained by combining the "approximate" notion with some other notion of generalized derivative. Thus, various authors have discussed the approximate Peano derivative as well as the approximate symmetric derivative. We shall not develop any details of these derivatives. Instead, we refer the interested reader to the recent works [2], [33], [35], [36], [37], and [48].

In the preceding sections we focused on approximate derivatives. We now wish to say something about approximately differentiable functions, that is, functions F for which F'_{ap} exists and is finite on some interval I. Such a function possesses a good deal of structure. It must, of course, be approximately continuous. While it need not be continuous, it must possess some properties of continuous functions. For example, it must achieve relative (though not absolute) extrema on closed intervals and it is determined by its values on a dense set [54]. Furthermore, it must be continuous on a dense, open set (because $F'_{ap} = F'$ on such a set) and, more generally, there must be a sequence of closed sets $\{E_k\}$ such that $I = \cup R_k$ and $F|E_k$ is continuous for all k [53].

Suppose, now, that F is a measurable function which is approximately differentiable at each point of some perfect set P. Suppose, further, that F vanishes on P. Must there be points of P at which F'_{ap} vanishes? If x is a point of density of P, then it is clear that $F'_{ap}(x) = 0$. But what if P is of measure zero and therefore has no points of density? Surprisingly, there still must be points at which F'_{ap} vanishes. In fact, the set $S \equiv \{x \in P: F'_{ap}(x) = 0\}$ must contain a dense open subset of P. This is a special case of the following result of O'Malley [53]: If F and G are measurable functions and P is a non-empty perfect set such that F = G on P, F is approximately differentiable on P and G is differentiable on P, then $F'_{ap} = G$ on a set containing a dense, open subset of P. (There are no additional restrictions on P - it need not have measure zero and it need not be nowhere dense.)

We end with a brief discussion of two topologies on the lines. The density topology (d-topology) consists of those sets A with the property that if x is in A, then x is a point of density of A. It is the coarsest topology relative to which all approximately continuous functions are continuous. A function is differentiable relative to the d-topology if and only if it is approximately differentiable. Which topology is the coarsest for which each approximately differentiable function is continuous? One might conjecture that this, too, is the d-topology. It is not! In [51], O'Malley showed that a still coarser topology, which he called the r-topology, is the right one. We shall not develop details of this topology here. For such details, and for several references to work on the d-topology, we refer the reader to [51].

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