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The Peano Curve and the Density Topology

In this paper we shall show that the component of the Peano curve serves as an example of density-to-density continuous function, which maps some sets of measure zero to sets with positive measure. This solves negatively the problem posed in [1].

The m-dimensional outer Lebesgue measure will be denoted by λ_m^*.

There are several equivalent definitions of the density topology d_m. We choose the following one convenient for us: A set $D \subset R^m$ will be termed open in (R^m, d_m), if for every $x \in D$ and $\varepsilon > 0$ there is a $h > 0$ such that

$$\lambda_m^* (B - D) < \varepsilon \lambda_m^* B$$

holds for every m-dimensional ball B with center at x and radius less than h.

Proposition 1. Let $f\colon [0, 1] \to R^2$ be a mapping with $\lambda_1^* A \le \lambda_2^* f(A)$ for every $A \subset [0, 1]$. Assume there is a $c > 0$ such that $|f(s) - f(t)| \le c|s-t|^{1/2}$ for every $s, t \in [0, 1]$ (the so called 1/2-Hölder-continuity condition). Then f is continuous from (R^1, d_1) to (R^2, d_2).

Proof. Let $U \subset R^2$ be open in (R^2, d_2). We shall prove that $G = f^{-1}(U)$ is open in the topology induced on [0, 1] by (R^1, d_1). Fix $t \in (0, 1) \cap G$ and $\varepsilon > 0$.

Find $h_0 > 0$ such that

$$\lambda_2^*(B - U) \leq 2\pi^{-1}c^{-2}\varepsilon\lambda_2^*B = 2h^2c^{-2}\varepsilon$$

holds for every ball B with center at f(t) and radius h less than h_0. Choose v with $0 < v < h_0^2c^{-2}$ and $[t-v, t+v] \subset [0, 1]$. Set $h = (vc^2)^{1/2}$, $I = [t-v, t+v]$, $B = \{z \in R^2: |z - f(t)| \leq h$. If $s \in I$, then

$$|f(s) - f(t)| \leq c|s-t|^{1/2} \leq chc^{-1} = h .$$

Hence $f(I - G) \subset B - U$ and we have

$$\lambda_1^*(I-G) \leq \lambda_2^* f(I-G) \leq \lambda_2^*(B-U) \leq$$

$$\leq 2h^2c^{-2}\varepsilon = 2v\varepsilon = \varepsilon\lambda_1^* I .$$

The same arguments prove the right and left density-to-density continuity at 0 and 1, respectively.

The existence of a mapping satisfying the hypotheses of Proposition 1 is not obvious. However, we shall show that the curve constructed by G. Peano in 1890 as the example of a continuous mapping of the interval [0, 1] onto the square $[0, 1]^2$ has the required properties. We may realize this curve as the limit f of the sequence $\{f_n = (f_n^1, f_n^2)\}$ defined as follows. Denote by $g = (g^1, g^2)$ the curve defined on [0, 9], which is linear on each of the intervals $[k-1, k]$ where $k = 1,2,\dots,9$ and which takes the values

$$g(0)=(0, 0), \qquad g(1)=(1, 1),$$
$$g(2)=(0, 2), \qquad g(3)=(1, 3),$$
$$g(4)=(2, 2), \qquad g(5)=(1, 1),$$
$$g(6)=(2, 0), \qquad g(7)=(3, 1),$$
$$g(8)=(2, 2), \qquad g(9)=(3, 3).$$

Put

$$f_1^i(t) = \frac{1}{3}\, g^i\ (9^{-1}t)$$

for $t \in [0, 1]$ and $i = 1,2$ and by induction

$$f_{n+1}^i(k\cdot 9^{-n}+t) = f_n^i(k\cdot 9^{-n}) \ +$$
$$+ \frac{1}{3}\, g^i(9^{n+1}t)\cdot[f_n^i((k+1)\cdot 9^{-n})-f_n^i(k\cdot 9^{-n})]$$

for $t \in [0, 9^{-n}]$; $k = 0,1,\dots,9^n-1$; $i = 1,2$. Denote by $\underline{I}$ the set of all the intervals $[(k-1)\cdot 9^{-n},(k+1)\cdot 9^{-n}]$ where $n \in N$, $k = 1,2,\dots,9^{n-1}$. It is obvious that

(1) If $I \in \underline{I}$ is an interval of the length $2\cdot 9^{-n}$, then $f(I)$ is an rectangle with the area $2\cdot 9^{-n}$ and the diameter $\sqrt{5}\ .\ 3^{-n}$.

Let $s,\ t \in [0,1]$, $9^{-n-1} < |s-t| \le 9^{-n}$. Then by (1) $|f(s)-f(t)| \le \sqrt{5} \cdot 3^{-n} < 3\cdot\sqrt{5}\ |s-t|^{1/2}$. Thus we have

(2) The curve f satisfies the 1/2-Hölder-continutiy condition.

We introduce a "new" outer measure $\tilde{\lambda}$ on $[0, 1]$ by $\tilde{\lambda}\ A = \lambda_2^* f(A)$. The assertion (1) shows $\tilde{\lambda} I = \lambda_1^* I$ for and $I \in \underline{I}$. It easily follows that $\tilde{\lambda} G = \lambda_1^* G$ holds for every G relatively open in [0, 1]. Let $U \subset [0, 1]^2$ be relatively open, $f(A) \subset U$. Then there is a relatively open subset G of [0, 1] with $A \subset G$ and $f(G) \subset U$, namely $G = f^{-1}(U)$. Denote by $\underline{G}$, $\underline{U}$ the family of all relatively open sets containing A, f(a), respectively.

Then we have

$$\lambda_1^* A = \inf_{\underline{G}} \lambda_1^* G = \inf_{\underline{G}} \tilde{\lambda} G = \inf_{\underline{G}} \lambda_2^* f(G) \leq$$

$$\inf_{\underline{U}} \lambda_2^* U = \lambda_2^* f(A) = \tilde{\lambda} A \quad \inf_{\underline{G}} \tilde{\lambda} G = \lambda_1^* A \quad .$$

We have obtained

(3) If $A \subset [0, 1]$, then $\lambda_1^* A = \lambda_2^* f(A)$.

Now we can prove that $F: =f^1$ is an example of a continuous mapping of (R^1, d_1) to itself which maps some sets of measure zero to sets with positive measure.

Obviously the projection mapping $P: R^2 \to R^1$ defined by $P((x^1, x^2)) = x^1$ is continuous from (R^2, d_2) to (R^1, d_1). Proposition 1 and the properties (2), (3) prove the continuity of f from (R^1, d_1) to (R^2, d_2). Hence the superposition $F = P \circ f$ is continuous from (R^1, d_1) to itself. Put $M = [0, 1] \times \{0\}$, $A = f^{-1}(M)$. By (3), $\lambda_1^* A = \lambda_2^* M = 0$. However, $F(A) = [0, 1]$ and thus F(A) has positive measure.

References

[1] Real Analysis Exchange 1 № 1 (1976), p.63.