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Ryszard Jerzy Pawlak, Institute of Mathematics, Łódź University,
ul. Banacha 22, 90-238 Łódź, Poland.

On continuity and monotonicity of restrictions of connected functions.

§0. Introduction, basic definitions, and notations.

This article contains results dealing with restrictions of real connected functions defined on some topological spaces. These results presented in the work [11] form extensions of researches of K. M. Garg [2], Z. Grande [4] and J. S. Lipinski [8].

We use the standard notions and denotations that were used in the monograph of R. Engelking [1] or in the article of K. M. Garg [3]. The notions that were not defined in [1] or [3] or those we define in a different way analogous to the ones defined by R. Engelking and K. M. Garg are defined immediately before we use them.

By R we shall denote the set of real numbers with its natural topology.

The symbol E^d denotes the derived set of a set E and closure and interior of a set A we denote by $\overline{A}$ and Int A. According to the notations in [3] we denote:

$$Y_c(f) = \{\alpha \in Y:\ f^{-1}(\alpha) \text{ is a connected set}\},$$

$$S_c(f) = f^{-1}(Y_c(f)),$$

$$S^c(f) = f^{-1}(\overline{Y_c(f)})$$

for a function $f:X \to Y$.

At last it is necessary to agree (according to the terminology of R. Engelking) that a compact space is always a Hausdorff space in which every open cover has a finite open subcover.

§ 1. Remarks on continuity, quasi-continuity and Blumberg sets for connected functions.

We say that a function $f:X \to Y$, where X,Y are arbitrary topological spaces, is connected if f(C) is a connected set for each connected set $C \subset X$.

Theorem 1.1. Let $f:X \to R$ be a connected function, where X is a connected and locally connected space. If $S \subset X$ fulfils the following conditions:

1^0 $f(S) \subset \overline{Y_c(f)}$

2^0 $f^{-1}(\alpha)$ is closed in S for every $\alpha \in f(S)$;

then $f_{|S}$ is continuous.

This theorem is essentially stronger than Garg's theorem. This result is a starting point of proving new theorems such as theorem 1.4 anc corollary 3.1.

Now theorem 1.1 suggests a problem: Is a restriction to $S^c(f)$ of a connected function f defined on a connected and locally connected space continuous? The answer to this problem is negative, as we can see in the follwoing example: Let X = R and $f:R \to R$ be as follows:

$$f(x) = \begin{cases} -x+1 & \text{for } x \in (-\infty, 0], \\ \sin \frac{1}{x} & \text{for } x \in (0, \frac{2}{3\pi}], \\ -\frac{3\pi}{2} x & \text{for } x \in (\frac{2}{3\pi}, \infty). \end{cases}$$

It is easy to see that $f_{|S^c(f)}$ is not a continuous function.

There is another question related to theorem 1.1: What kind of structure has the set P consisting of those elements of $\overline{Y_c(f)}$ for which $f^{-1}(\alpha)$ is not closed in $S_c(f)$? Theorem 1.2 describes this set.

Theorem 1.2. Let $f:X \to R$ is a connected function, where X is connected and locally connected space. Then

(i) if α is such a point of $\overline{Y_c(f)}$ that $f^{-1}(\alpha)$ is not closed in $S^c(f)$ then α is unilateral limit point of $Y_c(f)$;

(ii) the set P of all α from $\overline{Y_c(f)}$ for which $f^{-1}(\alpha)$ is not closed in $S^c(f)$ is at most denumerable.

The function $f: X \to R$ is called quasi-continuous at x (see [6] and [9]) if for every neighborhood W of f(x) and every neighborhood U of x the set Int $[U \cap f^{-1}(W)]$ is nonempty. The function $f:X \to R$ is called quasi-continuous if it is quasi-continuous at every point $x \in X$.

Theorem 1.3. Let $f:X \to R$ be a connected function defined on a connected and locally connected space X. Assume that $S \subset X$ fulfills the following conditions:

1^o $S_c(f) \subset S \subset S^c(f)$,

2^o for $x \in S$ if $x \in [S\ S_c(f)]^d$
then $x \in [S_c(f)\ \ f^{-1}(f(x))]^d$.

Then $f_{|S}$ is quasi-continuous.

Let X, Y be the topological spaces and $f: X \to Y$. We say that $B \subset X$ is a Blumberg set for f (see [9]) if B is dense in X and $f_{|B}$ is a continuous function.

Theorem 1.4. If $f: X \to R$ is a connected function defined on a connected and locally connected space X, then the function $f_{|S^c(f)}$ has a Blumberg set.

The Blumberg set $B \subset X$ is said to be a strong set of Blumberg for f if for each open set $V \subset X$ the set $f(V \cap B)$ is dense in $f(v)$.

The simple example shows that there exist such connected functions f defined on the connected and locally connected space, such that $f_{|S^c(f)}$ has no strong set of Blumberg. This example suggests the following question: What additional assumptions are sufficient for $f_{|S^c(f)}$ to have a strong set of Blumberg? Here is the answer.

Theorem 1.5. Let X be connected and locally connected space and $f: X \to R$ a connected function. Then the function $g = f_{|S^c(f)}$ has a strong set of Blumberg if and only if g is quasi-continuous.

§ 2. Monotonicity of connected functions.

We say that a non-void set K cuts a topological space X if $X\ K=A\cup B$, where A and B are non-void open and disjoint sets and we say that a function $f:X \to Y$ cuts the space X if each component of its arbitrary level $f^{-1}(y)$ cuts X.

Theorem 2.1. If $f:R^n \to R$ is a connected function cutting R^n, then the set $Y_c(f)$ is bilaterally closed (i.e. contains all of its bilateral accumulation points).

We say that a function $f:X \to Y$ is weakly monotone if every level is a connected set.

The above presented theorem suggests a question: Is $Y_c(f)$ closed under the assumptions of theorem 2.1? Moreover if the answer is "no" then: Is $f_{|\overline{S_c(f)}}$ weakly monotone? These questions are interesting (in view of Garg's theorems [2]) in the case where the domain of f is equal to R^n, $n \geq 2$.

The next theorem solves these problems.

Theorem 2.2(i) There exists a connected function $f:R^n \to R$ $(n=1,2,...)$ cutting R^n for which $Y_c(f)$ is not closed.

(ii) There exists a connected function $f:R^n \to R$ $(n=2,3,...)$ cutting R^n, for which $f_{|\overline{S_c(f)}}$ is not weakly monotone.

We say that a function $f:X \to Y$ strongly cuts the space X if:

1° f cuts the space X,

2° if $\alpha \in Y$ and S_1, S_2, S_3 are arbitrary distinct components of $f^{-1}(\alpha)$ and $C_1, C_2, C_3 \subset X$ are connected sets such that $C_i \cap S_j \neq \emptyset$ for $i \neq j$, then $C_i \cap S_i \neq \emptyset$ for some $i = 1,2,3$.

Theorem 2.3. For a connected function $f:R \to R$ the following conditions are equivalent:

(a) f cuts R,

(b) f strongly cuts R,

(c) no level of f contains a half-line.

Theorem 2.4. Let X be a regular connected and locally connected space, and $f:X \to R$ a connected function strongly cutting X. If α is a unilateral accumulation point of $Y_c(f)$, then the set $\overline{S_c(f)}$ meets at most two components of $f^{-1}(\alpha)$.

Now we can formulate two theorems giving an answer to the problem of K. M. Garg [3] (Problem 3.11 p.27). Partial resolutions are contained in articles [4] and [10]. Our results are a little more general with regard to the range of the considered functions. At the same time the assumptions of these theorems cannot be weakened as in analogous Grande theorems for real functions. Before writing the theorems we give the following definitions.

The function $f:X \to Y$ is said to be Morrey monotone (quasi-monotone) if each of its levels is a continuum ($f^{-1}(C)$ is connected for every connected set $C \subset f(X)$).

We say that a topological space X is connectedly embedded in a space Y if there is a connected injection $f:X \to Y$.

Theorem 2.5. If X is a locally connected continuum, Y connectedly embedded in R, then a connected function $f:X \to Y$ is Morrey monotone on the set $\overline{S_c(f)}$.

Theorem 2.6. If X is a locally connected continuum, Y connectedly embedded in R and $f:X \to Y$ is a connected function, then $f_{|\overline{S_c(f)}}$ is quasi-monotone.

§ 3. Properties of open connected functions.

Theorem 3.1. Let $f:X \to R$ be an open connected function on a connected and locally connected space X. Then the set $f^{-1}(\alpha)$ is bounded and closed for every $\alpha \in \overline{Y_c(f)}$.

Corollary 3.1. Let X be a connected and locally connected space. If $f:X \to R$ is an open connected function, then $f_{|S^c(f)}$ is continuous.

Theorem 3.2. Let X be a compact space and Y a Hausdorff space. If $f:X \to Y$ is open and continuous on $\overline{S_c(f)}$, then f is Morrey monotone on the set $\overline{S^c(f)}$.

Corollary 3.2. If X is a locally connected continuum and $f:X \to R$ is an open connected function, then f is Morrey monotone on the set $S^c(f)$.

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