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Intersections of Qualitative Cluster Sets

§1.

Let H denote the open upper half plane above the real line R and let z and x denote points in H and R respectively. Let $\overline{A}$ denote the closure of the set A. For two directions θ_1 and θ_2, $0 < \theta_1 < \theta_2 < \pi$, we write

$$\sigma_{\theta_1\theta_2} = \{z: z \in H; \theta_1 < \arg z < \theta_2\}$$

Then $\sigma_{\theta_1\theta_2}$ is a sector in H with vertex at the origin. By $\sigma_{\theta_1\theta_2}(x)$ we denote the translate of $\sigma_{\theta_1\theta_2}$ which is obtained by taking the origin at x. If there is no confusion then we simply write σ and $\sigma(x)$ instead of $\sigma_{\theta_1\theta_2}$ and $\sigma_{\theta_1\theta_2}(x)$. For a fixed $x \in R$, fixed $\theta \in (0,\pi)$, and $r > 0$ we write

$$L_\theta(x) = \{z: z \in H; \arg(z-x) = \theta\},$$

$$K(x,r) = \{z: z \in H; |z - x| < r\},$$

$$\sigma(x,r) = \sigma(x) \cap K(x,r), \text{ and}$$

$$L_\theta(x,r) = L_\theta(x) \cap K(x,r) .$$

Let $f:H \to W$ where W is a topological space. Then the qualitative cluster set $C_q(f,x)$ of f at x is the set of all $\omega \in W$ such that for every open set U in W containing ω, the set $f^{-1}(U) \cap K(x,r)$ is of the second category for each $r > 0$. The sectorial qualitative cluster set $C_q(f,x,\sigma)$ of f at x in the sector σ is the set of all $\omega \in W$ such that for every open set U in W containing ω, the set $f^{-1}(U) \cap \sigma(x,r)$ is of the second category for each $r > 0$. The directional qualitative cluster set $C_q(f,x,\theta)$ of f at x in the direction $\theta \in (0,\pi)$ is the set of all $\omega \in W$ such that for every open set U in W containing ω, $f^{-1}(U) \cap L_\theta(x,r)$ is of the second category in $L_\theta(x)$ for every $r > 0$. The directional cluster set $C(f,x,\theta)$ of f at x in the direction $\theta \in (0,\pi)$ is the set of all $\omega \in W$ such that for every open set U in W containing ω, x is a limit point of the set $f^{-1}(U) \cap L_\theta(x)$.

A set $E \subset H$ is said to have Baire property if there is an open set G and a first category set Q in H such that

$$E = G \,\Delta\, Q \equiv (G \setminus Q) \cup (Q \setminus G)$$

A function $f:H \to W$, where W is a topological space, is said to have Baire property if for every open set U in W the set $f^{-1}(U)$ has Baire property.

Jarnik [3] proved that if $f: H \to \Omega$ is arbitrary, where Ω is the extended real line, then the set of all points x in R at which there exit two directions θ_1 and θ_2 such that

$$C(f,x,\theta_1) \cap C(f,x,\theta_2) = \emptyset$$

is countable.

In this paper we shall prove an analogue of Jarnik's result for directional qualitative cluster sets. We also prove a result on intersection of sectorial qualitative cluster sets. This result is an analogue of the result proved in [1] by Allen and Belna on intersections of sectorial essential cluster sets. We would also like to mention that other interesting results on qualitative cluster sets can be found in [2] and [5].

The results which we proved here are the following:

(1) If $f: H \to W$ has the Baire property, where W is a compact, second countable, Hausdorff topological space then the set of all points x in R at which there exist a second category set $\Theta(x)$ of directions θ, $0 < \theta < \pi$, with the property that for each pair of directions $\{\theta, \varphi\} \subset \Theta(x)$

$$C_q(f, x, \theta) \cap C_q(f, x, \varphi) = \emptyset$$

is countable.

(2) *If $f:H \to W$ is arbitrary, where W is a compact, second countable, Hausdorff topological space, then the set of all points x in R at which there exist two sectors $\sigma^1(x)$ and $\sigma^2(x)$ in H such that*

$$C_q(f,x,\sigma^1) \cap C_q(f,x,\sigma^2) = \emptyset$$

is countable.

§3.

Before proving some auxiliary lemmas we shall define the following sets and relations among them which will be used in the sequel.

Let $E \subset H$ have Baire property. Then there is an open set G and a first category set Q in H such that $E = G \Delta Q$. For fixed $x \in R$ we write,

$$I(E,x) = \{\theta \in (0,\pi): L_\theta(x,r) \cap E \text{ is residual in } L_\theta(x,r) \text{ for at least one } r>0\}$$

$$\mathcal{G}(E,x) = \{\theta \in (0,\pi): L_\theta(x,r) \cap E \text{ is first category in } L_\theta(x,r) \text{ for at least one } r>0\}.$$

For positive integers m and n we also define

$$I_n(G,x) = \{\theta \in (0,\pi): \ L_\theta(x,1/n) \cap G \text{ is residual in } L_\theta(x,1/n)\},$$

and

$$\Theta_m(G,x) = \{\theta \in (0,\pi): \ L_\theta(x,1/m) \cap G \text{ is first category in } L_\theta(x,1/m)\} \ .$$

Then, since Q is of the first category, it follows from the Kurotowski-Ulam Theorem [4,p.56] that

$$\{\theta \in (0,\pi): \ L_\theta(x,r) \cap Q \text{ is of second category in } L_\theta(x,r)\}$$

is of the first category in $(0,\pi)$ for each $r > 0$. Since $E = G \,\Delta\, Q$, it follows that $I(E,x) \setminus I(G \setminus Q,x)$ and consequently, $I(E,x) \setminus I(G,x)$ is of first category. But, $I(G,x) \subset \bigcup_{n=1}^{\infty} I_n(G,x)$ and hence $I(E,x) \setminus \bigcup_{n=1}^{\infty} I_n(G,x)$ is also a first category set. Similarily, it can be shown that $\Theta(E,x) \bigcup_{n=1}^{\infty} \Theta_m(G,x)$ is a first category set. Now, let

$$P(E) = \{x \in R: \ \text{both } I(E,x) \text{ and } \Theta(E,x) \text{ are of second category}\}$$

LEMMA 1. _If_ $E \subset H$ _has the Baire property, then_ $P(E)$ _is countable_.

Proof. Let $E = G \,\Delta\, Q$ where G is open and Q is of first category in H. For positive integers m, and n, and rationals $0 < \alpha < \beta < \pi$, let

$$P_{mn\alpha\beta}(E) = \{x \in R\colon I_n(G,x) \neq \emptyset \text{ and } \Theta_m(G,x) \text{ is dense in } (\alpha,\beta)\}.$$

Using (1) and (2) it is easily seen that $P(E) \subset \bigcup P_{mn\alpha\beta}(E)$ where the union is taken over all positive integers m and n, and all rational numbers $0 < \alpha < \beta < \pi$. We show that each set $P_{mn\alpha\beta}(E)$ contains no two sided limit points and consequently is countable. Let m, n, α, and β be fixed, and let $P = P_{mn\alpha\beta}(E)$. As G is open and $\Theta_m(G,x)$ is dense on (α,β), it follows that $G \cap \sigma_{\alpha\beta}(x,1/m) \neq \emptyset$ for each $x \in P$. Now suppose that P contains a two sided limit point, x_0, of P. Then there is a sequence $\{x_i\} \subset P$ such that $x_{2i} > x_0$, $x_{2i+1} < x_0$, and $\lim_{i\to\infty} x_i = x_0$. As $x_0 \in P$, there is a $\theta_0 \in I_n(G,x_0)$ and either $\theta_0 < \alpha$ or $\beta < \theta_0$. But if $\theta_0 < \alpha$, there is a sufficiently large i such that $\sigma_{\alpha\beta}(x_{2i},1/m) \cap L_{\theta_0}(x_0,1/n) \neq \emptyset$ which contradicts the fact that G misses $\sigma_{\alpha\beta}(x_{2i}, 1/m)$ and is residual on $L_{\theta_0}(x_0, 1/n)$. If $\theta_0 > \beta$, there is a sufficiently large i such that $\sigma_{\alpha\beta}(x_{2i+1}, 1/m) \cap L_{\theta_0}(x_0, 1/n) \neq \emptyset$ and again, this is a contradiction.

Thus, each of the sets $P_{mn\alpha\beta}(E)$ is countable and consequently, $P(E)$ is countable.

Let $E \subset H$ and define

$$M(E) = \{x \in R: \text{ there are two sectors } \sigma^1(x) \text{ and } \sigma^2(x) \text{ such that for some } r > 0,\ \sigma^1(x,r) \cap E \text{ is residual in } \sigma^1(x,r), \text{ and } \sigma^2(x,r) \cap E \text{ is of the first category}\}.$$

LEMMA 2. _If_ $E \subset H$ _is arbitrary, then_ $M(E)$ _is countable._

Proof. Let $x \in M(E)$, then there is a natural number $n = n(x)$ and four rational directions $\alpha_i = \alpha_i(x)$, $\beta_i = \beta_i(x)$, $i=1,2$ such that

$$\sigma_{\alpha_1\beta_1}(x, 1/n) \cap E \text{ is residual in } \sigma_{\alpha_1\beta_1}(x,1/n),$$

and

$$\sigma_{\alpha_2\beta_2}(x,1/n) \cap E \text{ is first category.}$$

Let

$$M_{n\alpha_1\alpha_2\beta_1\beta_2} = \{x \in M(E): \ n=n(x), \alpha_i=\alpha_i(x), \beta_i=\beta_i(x) \text{ for } i=1,2\}.$$

We show that each set $M_{n\alpha_1\alpha_2\beta_1\beta_2}$ contains no two sided limit points, from which it follows that $M(E)$ is countable. Let $n, \alpha_1, \alpha_2, \beta_1, \beta_2$ be fixed and let

$M=M_{n\alpha_1\alpha_2\beta_1\beta_2}$. Now, as $\sigma_{\alpha_1\beta_1}(x) \cap \sigma_{\alpha_2\beta_2}(x) = \emptyset$

for each $x \in M$, it follows that either

$\alpha_1 < \beta_1 < \alpha_2 < \beta_2$ or $\alpha_2 < \beta_2 < \alpha_1 < \beta_1$

and for definiteness we suppose the former. Suppose x_0 is a two sided limit point of M. Then there is a sequence $\{x_i\} \subset M$ such that $x_{2i} > x_0$, $x_{2i+1} < x_0$, and $\lim_{i\to\infty} x_i = x_0$. However, for sufficiently large values of i it follows that $\sigma_{\alpha_1\beta_1}(x_{2i+1}, 1/n) \cap \sigma_{\alpha_2\beta_2}(x_0, 1/n) \neq \emptyset$ and this contradicts the fact that E is residual in $\sigma_{\alpha_2\beta_2}(x_0, 1/n)$ but of first category in $\sigma_{\alpha_1\beta_1}(x_{2i+1}, 1/n)$. If $\alpha_2 < \beta_2 < \alpha_1 < \beta_1$ a similar contradiction is reached using $\{x_{2i}\}$.

LEMMA 3. <u>If</u> $f:H \to W$ <u>is arbitrary, where</u> W <u>is a compact topological space and if</u> G <u>is an open subset of</u> W <u>such that</u> $C_q(f,x,\theta) \subset G$, <u>then there exists a positive integer</u> n <u>such that</u> $L_\theta(x,1/n) \cap f^{-1}(G)$ <u>is residual in</u> $L_\theta(x,1/n)$.

Proof. Since $W\setminus G$ is compact and disjoint from $C_q(f,x,\theta)$ there is a finite collection of open sets $\{V_i : i=1,2,\ldots,k\}$ and a corresponding set of radii $\{r_i > 0 : i=1,2,\ldots,k\}$ such that $W\setminus G \subset \bigcup_{i=1}^{k} V_i$ and

$f^{-1}(V_i) \cap L_\theta(x,r_i)$ is first category in $L_\theta(x,r_i)$ for each i. If $n = \min\{r_i : i = 1,2,\ldots k\}$, then $f^{-1}(\bigcup_{i=1}^{k} V_i) \cap L_\theta(x,1/n)$ is of first category in $L_\theta(x,1/n)$ and consequently, $f^{-1}(G) \cap L_\theta(x,1/n)$ is residual in $L_\theta(x,1/n)$.

LEMMA 4. _If_ $f:H\to W$ _is arbitrary where_ W _is a compact topological space, and if_ G _is an open subset of_ W _such that_ $C_q(f,x,\sigma) \subset G$, _then there is a positive integer_ n _such that_ $f^{-1}(G)\cap\sigma(x,1/n)$ _is residual in_ $\sigma(x,1/n)$.

Proof. The proof of LEMMA 4 is similar to that of LEMMA 3.

Suppose $f:H\to W$. In what follows, let $K=K(f)$ denote the set of all points $x\in R$ at which there is a second category set of directions $\Theta(x)$ with the property that if $\{\theta,\varphi\}\subseteq\Theta(x)$, then $C(f,x,\theta)\cap C(f,x,\varphi)=\emptyset$.

THEOREM 1. _Let_ W _be a compact, second countable, Hausdorff space. If_ $f:H\to W$ _has the Baire property, then_ K _is countable_.

Proof. Let $\mathcal{B}$ be a countable basis for the topology of W, let $\mathcal{A}$ be the countable collection of all sets G which can be expressed as a finite union of members of $\mathcal{B}$, and let $x_o \in K$. Then there is a second category set, $\Theta(x_o)$, of directions such that if $\{\theta,\varphi\}\subseteq\Theta(x_o)$, then $C(f,x,\theta)\cap C(f,x_o,\varphi) = \emptyset$. As $\Theta(x_o)$ is a second category subset of (o,π), there are two disjoint second category subsets, $\Theta_1(x_o)$ and $\Theta_2(x_o)$, of $\Theta(x_o)$. Let $\theta \in \Theta_1(x_o)$ be fixed. Then for every $\varphi \in \Theta_2(x_o)$ there is a $G \in \mathcal{B}$ such that $C_q(f, x_2, \theta) \subset G$ and $C_q(f, x_o, \varphi) \cap \overline{G} = \emptyset$. Since $\mathcal{B}$ is countable and $\Theta_2(x_o)$ is second category, there is a $G_\theta \in \mathcal{B}$ and a second category set $\Theta_2'(x_o) \subseteq \Theta_2(x_o)$ such that $C_q(f, x_o, \theta) \subset G_\theta$ and $C_q(f, x_o, \varphi) \cap \overline{G}_\theta = \emptyset$ for every $\varphi \in \Theta'_2(x_o)$; that is, $\Theta_2'(x_o) \subseteq \Theta(f^{-1}(G_\theta), x_o)$. By LEMMA 3 there is an n such that $L_\theta(x_o, 1/n) \cap f^{-1}(G_\theta)$ is residual in $L_\theta(x_o, 1/n)$, and hence, $\theta \in I(f^{-1}(G_\theta), x_o)$. But then, $\Theta_1(x_o) = \cup I(f^{-1}(G_\theta), x_o)$ where the union is taken over all $G \in \mathcal{B}$, and as $\Theta_1(x_o)$ is second category, there is a second category set $\Theta_1'(x_o) \subseteq \Theta_1(x_o)$ and a $G \in \mathcal{B}$ such that $G_\theta = G$ for every $\theta \in \Theta_1'(x_o)$. Consequently, both $I(f^{-1}(G), x_o)$ and $\Theta(f^{-1}(G), x_o)$ are second category and so $x_o \in P(f^{-1}(G))$. By LEMMA 1, each set $P(f^{-1}(G))$ is countable and as K is contained in a countable union of such sets, K itself is countable. This completes the proof of THEOREM 1.

COROLLARY. _Let_ W _be compact, second countable, and Hausdorff. If_ $f: H \to W$ _has the Baire property, then except for a countable set of points_ $x \in R$, _in every second category set of directions there is at least one pair of directions_, $\{\theta, \varphi\}$ such that $C_q(f, x, \theta) \cap C_q(f, x, \varphi) \neq \emptyset$.

Suppose $f:H \to W$. Let $M=M(f)$ denote the set of all points $x \in R$ at which there exist two sectors $\sigma^1(x)$ and $\sigma^2(x)$ in H such that $C_q(f,x,\sigma^1) \cap C_q(f,x,\sigma^2)=\emptyset$.

THEOREM 2. <u>Let W be a compact, second countable, Hausdorff space and let $f:H \to W$ be arbitrary. Then M is countable.</u>

Proof. Let $\mathcal{B}$ and $\mathcal{A}$ be the same as in THEOREM 1 and let $x_o \in M$. Then there exist sectors $\sigma^1(x_o)$ and $\sigma^2(x_o)$ such that $C_q(f,x_o,\sigma^1) \cap C_q(f,x_o,\sigma^2)=\emptyset$. Since $C_q(f,x_o,\sigma^1)$ and $C_q(f,x_o,\sigma^2)$ are disjoint closed sets and W is normal and compact, there is a $G_o \in \mathcal{A}$ such that $C_q(f,x_o,\sigma^1) \subset G_o$ and $C_q(f,x_o,\sigma^2) \cap \overline{G}_o = \emptyset$. Hence, using LEMMA 4 we find an n such that both $\sigma^1(x_o,1/n) \cap f^{-1}(G_o)$ is residual in $\sigma^1(x_o,1/n)$ and $\sigma^2(x_o,1/n) \cap f^{-1}(G_o)$ is first category; that is, $x_o \in M(f^{-1}(G_o))$. By LEMMA 2 each set $M(f^{-1}(G))$ is countable, and as M is contained in a countable union of such sets, M itself is countable. This completes the proof of THEOREM 2.

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