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Maximal Additive and Maximal Multiplicative Families for the Family of All Interval-Darboux Baire One Functions

1. In his book [1, p. 14] A. M. Bruckner defined the maximal additive and the maximal multiplicative family for a given family F of real functions as follows: A subfamily F_0 of the family F is called the maximal additive (multiplicative) family for F iff F_0 is the set of all functions f of F such that $f + g \in F$ ($fg \in F$) for all $g \in F$.

In [2 , Theorem 7.5, p. 109], A.M. Bruckner and J. G. Ceder proved that the maximal additive family for the family of all real Darboux functions of a real variable of the Baire class one is the family of all real continuous functions of a real variable.

In the cited book [1, p. 15] A.M. Bruckner gives the problem to find the maximal multiplicative family for the same family. R. Fleissner recently solved this problem in [3]. The maximal multiplicative family for the family of all real Darboux functions of a real variable of the Baire class one is the family of all real Darboux functions f of a real variable of the Baire class one having the following property:

If f is discontinuous from the right (from the left) at a point a, then f(a) = 0 and there exists a decreasing (an increasing) sequence $\{x_n\}_{n=1}^{\infty}$ converging to a such that $f(x_n) = 0$ for all n.

In [8], there are given the maximal additive and the maximal multiplicative family for the family of all real $\mathcal{B}$-Darboux Baire one functions defined on a finite dimensional strictly convex Banach space, where $\mathcal{B}$ is the base of all spherical neighborhoods. In this paper, we solve the problem of the maximal additive and maximal multiplicative family for the family of all real Interval-Darboux Baire one functions.

2. In [4], it is proved that a finite derivative of an additive Interval-function possesses the Darboux property in the strong sense on every interval. A real function f defined on the n-dimensional euclidean space E_n possesses the Darboux property in the strong sense on a closed interval I iff for every two points $p, q \in I$ and for each real number c such that $(f(p) - c)\cdot(f(q) - c) < 0$, there exists a point z from the interior of I such that f(z) = c.

In [9], C. J. Neugebauer introduced a class of some connected sets in E_n, called Darboux sets, and he said that a real function f defined on E_n possesses the Darboux property iff it maps every Darboux set into a connected set. In [6, p. 46] it is proved that a real function defined on E_n has the Darboux property in the sense of C. J. Neugebauer iff it possesses the

Darboux property in the strong sense on every closed interval.

We recall the definition of a $\mathcal{B}$-Darboux function. Let X be a topological space and let $\mathcal{B}$ be a base for the topology in X. In [5], there is given the following definition: A real function f defined on X is called $\mathcal{B}$-Darboux iff for each $A \in \mathcal{B}$, every $x, y \in \overline{A}$ ($\overline{A}$ denotes the closure of A) and each $c \in (\min (f(x), f(y)), \max (f(x),f(y)))$ there exists a point $z \in A$ such that $f(z)=c$. If X is E_n and $\mathcal{B}$ is the system of all open intervals in E_n, we shall call $\mathcal{B}$-Darboux functions Interval-Darboux functions. Interval-Darboux functions are functions which possess the Darboux property in the strong sense on every closed interval.

Let us recall the generalization of the theorem of Young for $\mathcal{B}$-Darboux functions:

Theorem 1. [7, Satz 9, p. 425] Let X be a complete metric space and let $\mathcal{B}$ be a base in X having the following two properties:

(1*) For each open neighborhood U of a point $x \in X$ and for each $B \in \mathcal{B}$ satisfying $x \in \overline{B}$ there exists a $C \in \mathcal{B}$ such that $C \subset U \cap B$ and $x \in \overline{C} - C$.

(2) For each $B \in \mathcal{B}$ and for each decomposition of B into two non empty disjoint sets A_1 and A_2 such that $\overline{U} \cap B \subset A_1$ and $\overline{U} \cap B \subset A_2$ respectively for each $U \in \mathcal{B}$ which is contained in A_1 and A_2, the sets $A_1' \cap A_2$ and $A_1 \cap A_2'$ are non empty (A_1' denotes the derived set of A_1).

Then a real Baire one function f defined on X is

$\mathcal{B}$ -Darboux iff for each $B \in \mathcal{B}$ and for each $x \in X$ satisfying $x \in \overline{B} - B$ there exists a simple sequence $\{x_n\}_{n=1}^{\infty}$ converging to x such that $x_n \in B$ for n = 1,2, 3,... and $\lim_{n \to \infty} f(x_n) = f(x)$.

We also recall that for $X = E_n$ and for the base of all open intervals in E_n the properties (1*) and (2) hold (see [5, pp. 47 - 48]).

3. Let $a_1 < b_1, \ldots, a_n < b_n$ and let $J = (a_1, \ldots, a_n; b_1, \ldots, b_n)$ be the open interval $\{(x_1, \ldots, x_n) \in E_n : a_i < x_i < b_i$ for $i = 1, 2, \ldots, n\}$. Let J_k and J_{k+1} respectively be the open intervals $(a_1 - \frac{1}{k}, \ldots, a_n - \frac{1}{k}; b_1 + \frac{1}{k}, \ldots b_n + \frac{1}{k})$ and $(a_1 - \frac{1}{k+1}, \ldots, a_n - \frac{1}{k+1}; b_1 + \frac{1}{k+1}, \ldots, b_n + \frac{1}{k+1})$.

Let φ_k be a bounded real continuous function on $E_n - J_k$ and let ψ_k be a real continuous function on $\overline{J}$. It is easy to see that there exists a continuous function X_k defined on $\overline{J}_{k+1}$ such that $X_k(A) = [-k-1, k+1]$ for each set $A = (\overline{J}_{k+1} - J_{k+1}) \cap B$, where $B \in \mathcal{B}$ is an open interval with the centre in $\overline{J}_{k+1} - J_{k+1}$ and the diameter (diam A) of the set A is not less than $\frac{1}{k+1}$. By the Tietze extension theorem, there exists a real bounded continuous function $\emptyset_k = \emptyset(\varphi_k, X_k, \psi_k)$ defined on E_n such that $\emptyset_k / E_n - J_k = \varphi_k$, $\emptyset_k / \overline{J}_{k+1} - J_{k+1} = X_k$, $\emptyset_k / \overline{J} = \psi_k$ and $\sup |\emptyset(\varphi_k X_k, \psi_k)| = \max(\sup |\varphi_k|, \sup |X_k|, \sup |\psi_k|)$.

We shall call a real function f defined on a closed interval $\overline{J}$, where J is an open interval, an Interval-Darboux function on $\overline{J}$ iff for each open interval I contained in J, for every x, y $\in \overline{I}$ and for each c $\in$(min (f(x), f(y)), max (f(x), f(y))) there exists a point z $\in$ I such that f(z) = c.

Lemma 1. (Extension lemma) Let J be an open interval in E_n and let f be a real Interval-Darboux Baire one function defined on $\overline{J}$. Then there exists a real Interval-Darboux Baire one function F defined on E_n such that $F/\overline{J} = f$.

Proof. Since f is a Baire one function on $\overline{J}$ there exists a sequence $\{\psi_k\}_{k=1}^{\infty}$ of real continuous functions defined on $\overline{J}$ such that $f(x) = \lim \psi_k(x)$ for each $x \in \overline{J}$.

Let J_k and X_k for k = 1, 2, 3, ... be such as above. Let c be a real number. Let φ_1 be a function defined on $E_n - J_1$ and $\varphi_1(x) = c$ for each $x \in E_n - J_1$. Let $\emptyset_1$ be a function $\emptyset(\varphi_1, X_1, \psi_1)$. By the induction, and using the function $\emptyset_k$, we define $\emptyset_{k+1}$ as follows: Let φ_{k+1} be the restriction of $\emptyset_k$ to $E_n - J_{k+1}$. Then $\emptyset_{k+1}$ is a function $\emptyset(\varphi_{k+1}, X_{k+1}, \psi_{k+1})$. The sequence $\{\emptyset_k\}_{k=1}^{\infty}$ is a sequence of bounded continuous functions defined on E_n. It is easy to prove that this sequence converges at each point $x \in E_n$. Let F be the pointwise limit of $\{\emptyset_k\}_{k=1}^{\infty}$. Then $F/\overline{J} = \lim_{k\to\infty} \psi_k = f$ and $F/E_n-\overline{J}$ is continuous function. The function F is a Baire one function on E_n.

Let I be an open interval and let $x \in E_n$ be a point for which $x \in \overline{I} - I$. If $x \in E_n - \overline{J}$, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points in I converging to x and $\lim_{n\to\infty} F(x_n) = F(x)$ since $F/E_n-\overline{J}$ is continuous. If $I \subset J$, then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points in I converging to x and $\lim_{n\to\infty} F(x_n) = F(x)$ since $F/\overline{J} = f$ and f is a real Interval-Darboux Baire one function on $\overline{J}$. If $x \in \overline{J} - J$ and $I \cap (E_n-\overline{J}) \neq \emptyset$, then there also exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points in I converging to x and $\lim_{n\to\infty} F(x_n) = F(x)$ since, according to the definition of the sequence $\{X_K\}_{k=1}^{\infty}$, F maps every neighborhood of x in $E_n - \overline{J}$ onto $(-\infty,\infty)$. From Theorem 1, it follows that F is an Interval-Darboux function and the lemma is proved.

4. Theorem 2. (Maximal additive family for the family of all real Interval-Darboux Baire one functions) The maximal additive family A for the family of all Interval-Darboux Baire one functions defined on E_n is the family of all real continuous functions defined on E_n.

Proof. Let f be a real continuous function defined on E_n. Then f + g is an Interval-Darboux Baire one function on E_n for each Interval-Darboux Baire one function g defined on E_n. This is a consequence of Theorem 13 of [7, Satz 13, p. 427]. Therefore $f \in A$.

Let f be a function which is discontinuous at a, $a \in E_n$. If f is not a real Interval-Darboux Baire one function on E_n, then f does not belong to A since

$f = f + 0$ is not a real Interval-Darboux Baire one function on E_n. If f is a real Interval-Darboux Baire one function on E_n, then it is evident that there exists an open interval I such that a is a vertex of I and $\alpha = \sup\{\inf f(J) :$ J is an open interval with one vertex a and which is contained in $I\} < \inf\{\sup f(J) :$ J is an open interval with one vertex a and which is contained in $I\} = \beta$. It is easy to prove that $\alpha \leq f(a) \leq \beta$. If we define g on $\overline{I}$ as follows: $g(x) = -f(x)$ for each $x \in \overline{I} - \{a\}$ and $g(a) \neq -f(a)$, $-\beta \leq g(a) \leq -\alpha$, then g is an Interval-Darboux Baire one function on $\overline{I}$. According to Lemma 1, there is a real Interval-Darboux Baire one function defined on E_n such that $G/\overline{I} = g$. The function $f + G$ is a real Baire one function on E_n, but it is not an Interval-Darboux function on E_n since $f(x) + G(x) = 0$ for each $x \in \overline{I} - \{a\}$ and $f(a) + G(a) \neq 0$. Therefore $f \notin A$.

5. Let f be a real function defined on E_n, let a be a point of E_n and let I be an open interval in E_n such that $a \in \overline{I} - I$. We shall say that a sequence $\{J_n\}_{n=1}^{\infty}$ of open intervals converges from an open interval I to a point a iff $\{J_n\}_{n=1}^{\infty}$ is decreasing sequence of open intervals contained in I, $a \in \overline{J}_n - J_n$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} \operatorname{diam} J_n = 0$. We shall say that f is discontinuous from I at a iff there exists a sequence $\{J_n\}_{n=1}^{\infty}$ of open intervals converging from I to a such that $\sup_n \inf f(J_n) < \inf_n \sup f(J_n)$.

Let M be the maximal multiplicative family for the family of all real Interval-Darboux Baire one functions defined on E_n.

Lemma 2. Let $f \in M$. Then $f^2 \in M$.

Proof. Let $f \in M$. Let g be a real Interval-Darboux Baire one function defined on E_n. Then fg is a real Interval-Darboux Baire one function defined on E_n since $f \in M$. Therefore also the function $f^2 g = f(fg)$ is a real Interval-Darboux Baire one function on E_n. Thus $f^2 \in M$.

Lemma 3. Let f be a nonnegative Interval-Darboux Baire one function defined on E_n. Let I be an open interval in E_n and $a \in E_n$ such that $a \in \overline{I} - I$. If f is discontinuous from I at a and $f(a) > 0$, then $f \notin M$.

Proof. Let f be a nonnegative Interval-Darboux Baire one function defined on E_n which is discontinuous from I at a. We can assume that a is a vertex of I. Then there exist two numbers α and β such that $\alpha = \sup_n \inf_n f(J_n) < \inf \sup_n f(J_n) = \beta$ for each sequence $\{J_n\}_{n=1}^{\infty}$ converging from I to a. Then $\alpha \leq f(a) \leq \beta$.

If $\alpha = f(a)$, then there exists an open interval J with one vertex a contained in I such that $f(J) \subset (\frac{\alpha}{2}, \infty)$. If $\alpha < f(a)$, we take a number γ from $(\alpha, f(a))$.

Now, we define a function g on $\overline{I}$ by: $g(x) = \frac{2}{\max(\alpha, 2f(x))}$ for each $x \in \overline{I} - \{a\}$ and $g(a) \in (\frac{1}{\beta}, \frac{1}{\alpha}) - \{\frac{1}{f(a)}\}$ if $\alpha = f(a)$ and $g(x) = \frac{2}{\max(\gamma, 2f(x))}$ for each

$x \in \overline{I} - \{a\}$ and $g(a) = \frac{1}{\gamma}$ if $\alpha < f(a)$. The function g is a real Interval-Darboux Baire one function on $\overline{I}$. By the extension lemma, there exists a real Interval-Darboux Baire one extension h of g defined on E_n. The function fh is not a real Interval-Darboux function on E_n since we have:

$(fh)(x) = 1$ for each $x \in \overline{J} - \{a\}$ and $(fh)(a) \neq 1$ if $\alpha = f(a)$ and

$(fh)(x) = 1$ for each $x \in \overline{I} - \{a\}$ which satisfies $\gamma \leq 2f(x)$, $(fh)(x) = \frac{2f(x)}{\gamma} < 1$ for each $x \in \overline{I} - \{a\}$ which satisfies $2f(x) < \gamma$ and $(fh)(a) = \frac{f(a)}{\gamma} > 1$ if $\alpha < f(a)$.

Therefore $f \notin M$.

Lemma 4. Let f be a nonnegative Interval-Darboux Baire one function defined on E_n which is discontinuous from an open interval I at a, $a \in E_n$. If $f(\overline{I} - \{a\}) \subset (0,\infty)$, then $f \notin M$.

Proof. Let f be as we assume in the lemma. Then we can assume that a is a vertex of I. Then there exist two numbers α and β such that $\alpha = \sup_n \inf f(J_n) < \inf_n \sup f(J_n) = \beta$ for each sequence $\{J_n\}_{n=1}^{\infty}$ converging from I to a. According to Lemma 3, $f \notin M$ if $f(a) > 0$. Let $f(a) = 0$. Then $\alpha = 0$. Let g be a function defined on $\overline{I}$ by :$g(x) = \frac{1}{f(x)}$ for each $x \in \overline{I} - \{a\}$ and $g(a) = \frac{1}{\min(1,\beta)}$. It is easy to prove that g is a real Interval-Darboux Baire one function on $\overline{I}$. By the extension lemma, there exists a real Interval-Darboux Baire one function h defined on E_n which extends g. The function fh is not a real Interval-Darboux function on E_n since $(fh)(x) = 1$ for each $x \in \overline{I} - \{a\}$ and $(fh)(a) = 0$.

Therefore $f \notin M$.

Lemma 5. Let f be a nonnegative Interval-Darboux Baire one function defined on E_n which is discontinuous from an open interval I at a, $a \in E_n$. Let $f(I) \subset (0,\infty)$. If f is discontinuous from every interval J at each point $z \in \overline{I}$ such that $f(z) = 0$, $J \subset I$ and $z \in \overline{J} - J$, then $f \notin M$.

Proof. According to Lemma 3, $f \notin M$ if $f(a) > 0$. Let $f(a) = 0$. Let $J = (a_1,\dots,a_n;b_1,\dots,b_n)$ be such an open interval for which $J \subset I$ and $\min f(\overline{J}) = 0$. Then we define a number $(f;J)$ as follows: $(f;J) = \max$ $\{i$: if $z = (z_1,\dots,z_n) \in \overline{J}$ is such that $f(z) = 0$, then the cardinal number of the set $\{j \in \{1, 2, \dots, n\}$: $z_j \notin \{a_j, b_j\}$ is at most $n - i\}$. Since $f(J) \subset (0,\infty)$, $J \subset I$ and $\min f(\overline{J}) = 0$, we have $(f;J) \geq 1$. Thus also $(f;I) \geq 1$.

It is easy to prove that for each $z \in \overline{I}$ satisfying $f(z) = 0$ there exists a positive number $\alpha(z)$ such that $\sup f(J) \geq \alpha(z)$ for each open interval which is contained in I and for which $z \in \overline{J} - J$.

If there exists an open interval $J = (a_1,\dots,a_n;$ $b_1,\dots,b_n)$ contained in I for which $(f;J) = n$, then $\emptyset \neq \{z \in \overline{J} : f(z) = 0\} \subset \{u = (u_1,\dots,u_n) \in \overline{J}$:

$u_i \in \{a_i, b_i\}$ for each $i = 1, 2, \ldots, n\}$. But, then the set $\{z \in \overline{J} : f(z) = 0\}$ is finite and therefore there exists an open interval Y contained in J such that f is discontinuous from Y at a point $y \in \overline{Y} - Y$ and $f(\overline{Y} - \{y\}) \subset (0, \infty)$. According to Lemma 4, $f \notin M$.

Let κ be a positive integer satisfying $n-1 \geq \kappa \geq (f;I)$ with the following property: If there exists such an open interval J contained in I for which $\min f(\overline{J}) = 0$ and $(f;J) \geq \kappa+1$, then f does not belong to M.

Now, suppose there exists an open interval J contained in I such that $(f;J) = \kappa$. We shall prove that f does not belong to M. Then there exists a point $z = (z_1, \ldots z_n) \in \overline{J}$, $f(z) = 0$ and $n-\kappa$ different positive intergers $i_1, \ldots, i_{n-\kappa}$ in $\{1, 2, \ldots, n\}$ such that $z_i \in \{a_i, b_i\}$ iff $i \in \{1, 2, \ldots, n\} - \{i_1, \ldots, i_{n-\kappa}\}$. Then there exists a positive number ϵ_0 such that $a_{i_s} < z_{i_s} - \epsilon_0 < z_{i_s} + \epsilon_0 < b_{i_s}$ for $s = 1, 2, \ldots, n-\kappa$. Let $0 < \epsilon \leq \epsilon_0$ and $B_\epsilon = \{(x_1, \ldots, x_n) \in \overline{J} : x_i = z_i$ for $i \in \{1, 2, \ldots, n\} - \{i_1, \ldots, i_{n-\kappa}\}$ and $z_{i_s} - \epsilon \leq x_{i_s} \leq z_{i_s} + \epsilon$ for $s = 1, 2, \ldots, n-\kappa\}$. We shall prove: If there exists an ϵ such that $0 < \epsilon \leq \epsilon_o$ and $f(B_\epsilon) = \{0\}$, then $f \notin M$.

Let $0 < \epsilon \leq \epsilon_0$ and $f(B_\epsilon) = \{0\}$. For n = 1, 2, 3, .. we put $B_n = \{v \in B : \alpha(v) \geq \frac{1}{n}\}$. Then we have: $B_\epsilon = \bigcup_{n=1}^{\infty} B_n$. Since B is of the second category of Baire in itself, there exists an n such that B_n is not nondense in B_ϵ.

Thus, for $s = 1, 2, \ldots, n-k$, there must exist numbers c_{i_s} and d_{i_s} such that $z_{i_s} - \epsilon \leq c_{i_s} < d_{i_s} \leq z_{i_s} + \epsilon$ for $s = 1, 2, \ldots, n-k$ and $C = \{v \in B_\epsilon : c_{i_s} \leq v_{i_s} \leq d_{i_s}$ for $s = 1, 2, \ldots, n-k\} \subset \overline{B}_n$. But, then for each $u \in C$, for each open interval Y which is contained in J and for which $u \in \overline{Y} - Y$ there exists a v in $B_n \cap \overline{Y}$. Therefore $0 = \inf f(Y) < \frac{1}{n} \leq \sup f(Y)$. This implies that f is discontinuous from Y at u. Thus $f(u) = 0$ and $\alpha(u) \geq \frac{1}{n}$ or according to Lemma 3, $f \notin M$. Therefore we can assume that $C \subset B_n$.

Let $\gamma_i \in (a_i, b_i)$ for $i \in \{1, 2, \ldots, n\} - \{i_1, \ldots, i_{n-k}\}$. Let Y be the open interval $\{(x_1, \ldots, x_n) \in E_n : \min(\gamma_i, z_i) < x_i < \max(\gamma_i, z_i)$ for $i \in \{1, 2, \ldots, n\} - \{i_1, \ldots, i_{n-k}\}$ and $c_{i_s} < x_{i_s} < d_{i_s}$ for $s = 1, 2, \ldots, n-k\}$. Let $t \in \overline{Y} - C$. Then $a_{i_s} < z_{i_s} - \epsilon \leq c_{i_s} \leq t_{i_s} \leq d_{i_s} \leq z_{i_s} + \epsilon < b_{i_s}$ for $s = 1, 2, \ldots, n-k$ and the set $\{i \in \{1, 2, \ldots, n\} - \{i_1, \ldots, i_{n-k}\} : t_i \neq z_i\}$ is nonempty. Therefore $t \in \overline{J}$ and the cardinal number of the set $\{i \in \{1, 2, \ldots, n\} : t_i \notin \{a_i, b_i\}\}$ is at least $n-k+1$. Thus $f(t) > 0$ since $(f;J) = k$. This implies that $\{v \in \overline{Y} : f(v) = 0\} = C$. Let g be a function defined by : $g(u) = \frac{1}{f(u)}$ for $u \in \overline{Y} - C$ and $g(u) = n$ for $u \in C$. From the generalization of the theorem of Young, we conclude that g is a real Interval-Darboux Baire one function on $\overline{Y}$. From the extension lemma, there exists a real Interval-Darboux Baire one function h defined on E_n which extends g. But, $f \notin M$ since $(fh)(u) = 1$ for each $u \in \overline{Y} - C$ and $(fh)(u) = 0$ for each $u \in C$.

If there does not exists an ϵ such that $f(B_\epsilon) = \{0\}$ and $0 < \epsilon \leq \epsilon_0$, then there exists a $w \in B_\eta - \{v \in \overline{J} : f(v) = 0\}$ for $\eta = \frac{1}{2}\epsilon_0$. Then $f(w) > 0$. According to Lemma 3, $f \notin M$ if there exists an open interval Y such that $Y \subset J$, $w \in \overline{Y} - Y$ and f is discontinuous from Y at w.

Let us assume that f is not discontinuous from any open interval Y at w such that $Y \subset J$ and $w \in \overline{Y} - Y$. Let $\sigma > 0$ and $B_{w,\sigma} = \{t \in B_\eta : w_{i_s} - \sigma < t_{i_s} < w_{i_s} + \sigma$ for $s = 1, 2, \ldots, n-k\}$. It is easy to prove that there exists a positive number σ such that $f(t) > \frac{f(w)}{2} > 0$ for each $t \in B_{w,\sigma}$. Let $W = \cup\{B_{w,\sigma} : \sigma > 0,\ f(B_{w,\sigma}) \subset (0,\infty)\}$. There exists such a positive number ω that $W = \{t \in \overline{I} : t_i = z_i$ for $i \in \{1, 2, \ldots, n\} - \{i_1, \ldots, i_{n-k}\}$ and $w_{i_s} - \omega < t_{i_s} < w_{i_s} + \omega$ for $s = 1, 2, \ldots, n-k\}$.

It is evident that $\omega \leq \max\{|w_{i_s} - z_{i_s}| : s = 1, 2, \ldots, n-k\} < \frac{1}{2}\epsilon_0$ and $f(W) \subset (0,\infty)$. Since $\overline{W} - W$ is compact, we have $\min f(\overline{W}) = 0$. It also holds: $a_{i_s} < z_{i_s} - \epsilon_0 < w_{i_s} - \omega < w_{i_s} + \omega < z_{i_s} + \epsilon_0 < b_{i_s}$ for $s = 1, 2, \ldots, n-k$.

Let $\gamma_i \in (a_i, b_i)$ and let $c_i = \min(z_i, \gamma_i)$, $d_i = \max(z_i, \gamma_i)$ for $i \in \{1, 2, \ldots, n\} \setminus \{i_1, \ldots, i_{n-k}\}$. Let

$c_{i_s} = w_{i_s} - \omega$, $d_{i_s} = w_{i_s} + \omega$ for $s = 1,2,\ldots, n-k$.

Let $Y = (c_1,\ldots,c_n;d_1,\ldots,d_n)$. Then $Y \subset J \subset I$.
Let $t \in \overline{Y}$ and $\{i \in \{1,2,\ldots,n\}-\{i_1,\ldots,i_{n-k}\}: t_i \neq z_i\} \neq \emptyset$. Then the cardinal number of the set $\{i \in \{1,2,\ldots,n\}: t_i \notin \{a_i,b_i\}\}$ is at least $n-k+1$. Since $t \in \overline{J}$ then $f(t)$ is a positive number. If f is such a point of $\overline{Y}$ for which $t_i = z_i$ for $i \in \{1, 2, \ldots, n\}-\{i_1, \ldots, i_{n-k}\}$ and $c_{i_s} = w_{i_s} - \omega < t_{i_s} < w_{i_s} + \omega = d_{i_s}$ for $s = 1, 2, \ldots, n-k$, then $t \in W$. Since $f(W) \subset (0,\infty)$, $f(t) > 0$. Therefore it must hold that $t \in \overline{W} - W$ for each $t \in \overline{Y}$ satisfying $f(t) = 0$. But, then the cardinal number of the set $\{i \in \{1, 2, \ldots,n\}: t_i \notin \{c_i, d_i\}\}$ is at most $n - (k+1)$ for each $t \in \overline{Y}$ satisfying $f(t) = 0$. Therefore $(f;Y) \geq k+1$. From the definition of the number k, we get that f does not belong to M.

From the induction principle, it is clear that we have proved the following : If there exists such an open interval J contained in I for which $(f;J) \geq 1$, then $f \notin M$. But, then $f \notin M$ since $(f;I) \geq 1$ and the lemma is proved.

Theorem 3. (Maximal multiplicative family for the family of all real Interval-Darboux Baire one functions) A real function f defined on E_n belongs to the maximal multiplicative family M for the family of all real Interval-Darboux Baire one functions defined on E_n iff f is a real Interval-Darboux Baire one function on E_n with the following property:

If f is discontinuous from an open interval I at a point a, then f(a) = 0 and there exists a sequence $\{a_n\}_{n=1}^{\infty}$ converging to a such that $f(a_n)=0$ for all n and either $a_n \in I$ for n = 1, 2, 3, ... or $a_n \in \overline{I} - I$ and there exists a sequence $\{I_n\}_{n=1}^{\infty}$ of open intervals contained in I such that f is not discontinuous from I_n at a_n.

Proof. Let f be a real Interval-Darboux Baire one function defined on E_n with the property mentioned in the theorem. Let g be a real Interval-Darboux Baire one function defined on E_n. Then fg is a Baire one function on E_n. To prove that fg is also an Interval-Darboux function on E_n, we use the generalization of the theorem of Young. Let $a \in E_n$, let I be an open interval such that $a \in \overline{I} - I$. If f is not discontinuous from I at a, then $\sup_n \inf f(J_n) = \inf_n \sup f(J_n)$ for each sequence $\{J_n\}_{n=1}^{\infty}$ of open intervals converging from I to a. Then it holds : $f(a) = \sup_u \inf f(J_n) = \inf_n \sup f(J_n)$ since f is an Interval-Darboux function on E_n. According to Theorem 1, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of points such that $x_n \in J_n$ and $g(x_n) \in (g(a) - \frac{1}{n}, g(a) + \frac{1}{n})$ for all n. We have : $f(a) = \sup_n \inf f(J_n) \leq \lim_{n\to\infty} \inf f(x_n) \leq \lim_{n\to\infty} \sup f(x_n) \leq \inf_n \sup f(J_n) = f(a)$. Therefore

$\lim_{n\to\infty} (f(x_n)g(x_n)) = f(a)g(a)$ and $x_n \in I$.

If f is discontinuous from I at a, then $f(a) = 0$ and there exists a sequence $\{a_n\}_{n=1}^{\infty}$ converging to a such that $f(a_n) = 0$ for all n, and either $a_n \in I$ for $n = 1, 2, \ldots$ or $a_n \in \overline{I} - I$ for $n = 1, 2, 3, \ldots$ and there exists a sequence $\{I_n\}_{n=1}^{\infty}$ of open intervals contained in I such that f is not discontinuous from I_n at a_n. In the first case, we have : $a_n \in I$ for $n = 1, 2, 3, \ldots$ and $\lim_{n\to\infty} (f(a_n)g(a_n)) = 0 = f(a)g(a)$. In the second case, we prove, as shown above, that for each $n = 1, 2, \ldots$ there exists a sequence $\{x_{n,k}\}_{k=1}^{\infty}$ of points of I_n such that $\lim_{k\to\infty} (f(x_{n,k})g(x_{n,k})) = f(a_n)g(a_n) = 0$. Let k_n be such a positive integer for which $\lim_{n\to\infty} x_{n,k_n} = a$ and $|f(x_{n,k_n})| < \frac{1}{n}$ for $n = 1, 2, 3, \ldots$. Then we have : $\{x_{n,k_n}\}_{n=1}^{\infty}$ is a sequence of points in I converging to a such that $\lim_{n\to\infty} (f(x_{n,k_n})g(x_{n,k_n})) = f(a)g(a)$. From the generalization of the theorem of Young, it follows that the function fg is an Interval-Darboux function.

So we have proved that $f \in M$.

Now, let $f \in M$. Since $f = f\cdot 1$, f is a real Interval-Darboux Baire one function. Let I be an open interval, $a \in E_n$ and $a \in \overline{I}-I$. Let f be discontinuous from I at a. Then f^2 is also discontinuous from I at a. According to Lemmas 2, 3, 4, and 5, $f^2(a) = 0$ and there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of points in $\overline{I}$ converging to a that $f^2(a_n) = 0$ for $n = 1, 2, 3, \ldots$, and either $a_n \in I$ for $n = 1, 2, 3, \ldots$ or $a_n \in \overline{I} - I$ for $n = 1, 2, 3, \ldots$ and there exists a sequence $\{J_n\}_{n=1}^{\infty}$ of open intervals contained in I such that f^2 is not discontinuous from J_n at a_n. Therefore $f(a) = 0$ and there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of points in $\overline{I}$ converging to a such that $f(a_n) = 0$ for $n = 1,2,3,\ldots$ and either $a_n \in I$ for $n = 1, 2, 3, \ldots$ or $a_n \in \overline{I} - I$ and there exists a sequence $\{J_n\}_{n=1}^{\infty}$ of open intervals contained in I such that f is not discontinuous from J_n at a_n.

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