

Jan Mařík, Department of Mathematics, Michigan State University, East Lansing, Michigan 48824.

LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER  
WITH DISCONTINUOUS COEFFICIENTS

Simple examples show that the basic theorems about the equation

$$(1) \quad y'' + ay' + by = 0,$$

where  $a$  and  $b$  are functions continuous on an interval, fail, if we drop the assumption of continuity. It seems to be of some interest to investigate conditions under which these theorems remain preserved. For this purpose it is not enough to assume that the solution space of (1) has dimension 2. Even in this case it may happen, for instance, that there is a point  $p$  in the interval in question such that  $u(p) = 0$  for each solution  $u$ . Theorem 9 of this paper shows, however, that the equation (1) "behaves normally", if it has two solutions whose Wronskian is nowhere zero. Let us say that such an equation is normal. The main goal of this paper is to investigate the "normality" of (1). Theorem 22 says that (1) is normal iff it is equivalent to an equation  $(Fy')' + G'y = 0$ , where  $F, G$  are differentiable and  $F$  is positive. This and 13 gives a necessary and sufficient condition for (1) to be normal under the assumption that there is a solution which is nowhere zero. The

result of 19 is analogous; we suppose there, however, that we have a solution whose derivative is nowhere zero.

1. Conventions and notations. Throughout this paper, an interval is always a one-dimensional non-degenerate interval. A function is a finite real function, a derivative a finite derivative. The symbol  $f'(x)$  will sometimes denote the derivative of  $f$  at  $x$  with respect to a given interval (so that  $f'(x)$  may be a one-sided derivative); similarly for  $f''(x)$ . It always will be obvious from the context which interval is meant. The meaning of a statement like " $F' = f$  on  $[0,1]$ " is now clear. If  $F, f$  are functions,  $I$  an interval and if  $F' = f$  on  $I$ , we say that  $F$  is a primitive of  $f$  (on  $I$ ).

In what follows,  $I$  is an interval and  $a, b$  are functions on  $I$ . For each function  $y$  twice differentiable on  $I$  we set  $Ly = y'' + ay' + by$ . If  $u, v$  are functions differentiable on  $I$ , then  $W_{u,v}$  means  $u'v - uv'$ . If  $f$  is a function on  $I$  and if  $J \subset I$ , then " $f \neq 0$  on  $J$ " means that  $f(x) \neq 0$  for each  $x \in J$ ; similarly for " $f > 0$  on  $J$ " etc.

2. Let  $u, v$  be twice differentiable on  $I$ . Then

$$W'_{u,v} + aW_{u,v} = vLu - uLv.$$

(Easy.)

3. Let  $v$  be twice differentiable,  $v \neq 0$  on  $I$ . Let  $V, F$  be functions such that  $V' + aV = 0$ ,  $v^2 F' = V$  on  $I$ . Define  $u = vF$ . Then  $W_{u,v} = V$ ,  $vLu = uLv$ .

Proof. We have  $W_{u,v} = v^2(u/v)' = v^2 F' = V$ ; by 2 we get  $vLu - uLv = V' + aV = 0$ .

4. Let  $v$  be twice differentiable,  $v' \neq 0$  on  $I$ . Let  $Q, T$  be functions such that  $Q'v' = bvQ$ ,  $T'v' = bQ$ . Define  $u = Tv - Q$ . Then  $W_{u,v} = Qv'$ ,  $Lu = T \cdot Lv$ .

Proof. Obviously  $T'v = Q'$  so that  $u' = Tv'$ ,  $W_{u,v} = Tv'v - Tvv' + Qv' = Qv'$ ,  $Lu = T'v' + Tv'' + aTv' + bTv - bQ = T \cdot Lv$ .

5. Let  $Lv = 0$ ,  $v' \neq 0$  on  $I$ . Let  $V, T$  be functions such that  $V' + aV = 0$ ,  $T'v'^2 = bV$ . Define  $u = Tv - V/v'$ . Then  $W_{u,v} = V$ ,  $Lu = 0$ .

Proof. Define  $Q$  by  $Qv' = V$ . Then  $0 = Q'v' + Qv'' + aQv' = Q'v' + Q \cdot (-bv)$ . Now we apply 4.

6. Let  $u, v$  be differentiable,  $|v| + |v'| > 0$ ,  $W_{u,v} = 0$  on  $I$ . Then  $u$  is a constant multiple of  $v$ .

Proof. Define  $c(x) = u(x)/v(x)$ , if  $v(x) \neq 0$ ,

and  $c(x) = u'(x)/v'(x)$ , if  $v(x) = 0$ . In the latter case we have  $u(x) = 0$  so that  $c$  is continuous and  $u = cv$  on  $I$ . Obviously  $c'(x) = 0$  wherever  $v(x) \neq 0$ . This easily implies that  $c$  is constant.

7. Let  $Lu = Lv = 0$ . We say that the functions  $u, v$  form a fundamental system of the equation

$$(1) \quad Ly = 0$$

iff  $W_{u,v} \neq 0$  on  $I$ . The equation (1) is called normal iff it has a fundamental system.

8. Let  $f, g$  be functions such that  $f' + af = g' + ag = 0$  and  $f \neq 0$  on  $I$ . Then  $g/f$  is constant.

Proof. Obviously  $(g/f)' = 0$ .

9. Let  $u, v$  form a fundamental system of (1) and let  $Lz = 0$ . Then there are numbers  $c, d$  such that  $z = cu + dv$  on  $I$ .

Proof. We have  $W_{u,v} \cdot z + W_{v,z} \cdot u + W_{z,u} \cdot v = 0$ . By 2 and 8 the functions  $W_{v,z}/W_{u,v}, W_{z,u}/W_{u,v}$  are constant.

10. Let  $a = A'$  on  $I$ . Let  $Lu = Lv = 0$ . Then  $e^A \cdot W_{u,v}$  is constant.

(Follows from 2 and 8.)

11. Let  $a$  have a primitive. Let  $u, v$  be

linearly independent solutions of (1) and let  $|v| + |v'| > 0$  on  $I$ . Then  $u, v$  form a fundamental system of (1).

(Follows from 10 and 6.)

12. Let  $a$  have a primitive,  $Lu = Lv = 0$ ,  $p \in I$ ,  $u(p) = u'(p) = 0$ . Suppose that there are  $x_n \in I$  such that  $x_n \neq p$ ,  $x_n \rightarrow p$ ,  $u(x_n) \neq 0$ . Then  $v(p) = v'(p) = 0$ .

Proof. Let  $|v(p)| + |v'(p)| > 0$ . There is a  $\delta > 0$  such that  $|v| + |v'| > 0$  on  $I \cap (p - \delta, p + \delta)$ . It follows from 11 that  $W_{u,v}(p) \neq 0$  - a contradiction.

13. Let  $a$  have a primitive. Let  $Lv = 0$ ,  $v \neq 0$  on  $I$ . Then (1) is normal.

Proof. We apply 3 with  $v = e^{-A}$ , where  $A' = a$ .

The proofs of the next two statements are left to the reader.

14. Let  $f$  be differentiable and let  $g$  be continuously differentiable on  $I$ . Then  $f'g$  has a primitive.

15. Let  $f$  be a function on  $I$ . Suppose that for each  $c \in I$  there is a  $\delta > 0$  such that  $f$  has a primitive on  $I \cap (c - \delta, c + \delta)$ . Then  $f$  has a primitive on  $I$ .

16. Let the equation (1) be normal. Then there are functions  $A, G$  such that  $A' = a$ ,  $G' = b \cdot e^A$  on  $I$ .

Proof. Let  $u, v$  form a fundamental system of (1); set  $A = -\log |W_{u,v}|$ . According to 2 we have  $A' = a$ . Now let  $c \in I$ . Let, e.g.,  $v(c) \neq 0$ . There is a  $\delta > 0$  such that  $v \neq 0$  on  $J = I \cap (c - \delta, c + \delta)$ . We have  $be^A = -(e^A v')'/v$  on  $J$ . Now we apply 14 and 15.

17. Let  $\Omega$  be an open set in the two-dimensional Euclidean space. Let  $H, h$  be functions continuous on  $\Omega$ . Suppose that, for each  $p \in \Omega$ ,  $h(p)$  is the partial derivative with respect to the first variable of  $H$  at  $p$ . Let  $f$  be a differentiable function on  $I$  such that  $\langle x, f(x) \rangle \in \Omega$  for each  $x \in I$ . Then the function  $f'(x) \cdot H(x, f(x))$  ( $x \in I$ ) has a primitive.

Proof. Let  $c \in I$ . There is a  $\delta > 0$  and intervals  $J_1, J_2$  with the following properties:  $J_1 = I \cap (c - \delta, c + \delta)$ ;  $f(J_1) \subset J_2$ ;  $J_1 \times J_2 \subset \Omega$ . For each  $x \in J_1$  set  $K(x) = \int_{f(c)}^{f(x)} H(x, t) dt$ ,  $q(x) = \int_{f(c)}^{f(x)} h(x, t) dt$ . Since  $q$  is continuous, there is a  $Q$  such that  $Q' = q$  on  $J_1$ . Obviously  $(K - Q)'(x) = f'(x) \cdot H(x, f(x))$  for each  $x \in J_1$ . Now we apply 15.

18. Let  $f, g$  be functions on  $I$ . Let  $f'$  exist and let  $g'$  be continuous on  $I$ . Let  $F$  be a function

continuously differentiable on an open interval  $J$ .

Let  $\varphi$  be a function such that  $\varphi' = f'g$  on  $I$  and that  $\varphi(I) \subseteq J$ . Then the function  $f'(x) \cdot F(\varphi(x))$  ( $x \in I$ ) has a primitive.

Proof. It is possible to extend the functions  $f, g, \varphi$  to an open interval  $I_1 \supset I$  in such a way that our requirements are fulfilled, even if  $I$  is replaced by  $I_1$ . Therefore, we may assume that  $I$  is open. Set  $\Psi = fg - \varphi$ ,  $\Omega = \{ \langle x, y \rangle; x \in I, g(x) \cdot y - \Psi(x) \in J \}$ . For  $\langle x, y \rangle \in \Omega$  define  $H(x, y) = F(g(x) \cdot y - \Psi(x))$ . Obviously  $H(x, f(x)) = F(\varphi(x))$  ( $x \in I$ ). Since  $F'$ ,  $g'$  and  $\Psi' = fg'$  are continuous, we may apply 17.

19. Let  $Lv = 0$  and let  $v' \neq 0$  on  $I$ . Then (1) is normal iff  $b/v'$  has a primitive.

Proof. I.) Let (1) be normal. By 16 there are  $A, G$  such that  $A' = a$ ,  $G' = b \cdot e^A$ . Set  $\varphi = v' \cdot e^A$ . Obviously  $\varphi' = -be^A v = -G'v$ ,  $b/v' = G'/\varphi$ . Now we apply 18 (with  $f = G$ ,  $F(t) = 1/t$  etc.).

II.) Let  $f' = b/v'$ . By 14 and 18 there are functions  $\varphi, T$  such that  $\varphi' = f'v$ ,  $T' = f' \cdot e^\varphi$ . Now we apply 4 (with  $Q = e^\varphi$ ).

20. Let  $I = (-1, 1)$ . Then there are functions  $a, b, b^*$  on  $I$  with the following properties:

1) Each of the functions  $a, b$  is continuous on  $I - \{0\}$  and has a primitive on  $I$ ;

- 2)  $b = b^*$  on  $I - \{0\}$ ;
- 3) the equation (1) is not normal;
- 4) the equation

$$(1^*) \quad y'' + ay' + b^*y = 0$$

is normal.

Proof. Set  $f(x) = x^3 \cos x^{-2}$ ,  $g(x) = x^3 \sin x^{-2}$  ( $x \neq 0$ ),  $f(0) = g(0) = 0$ ,  $K(x) = \frac{1}{2} (f(x)g(x) + 3x)$  ( $x \in \mathbb{R}$ ). It is easy to see that  $f$  and  $g$  are differentiable on  $\mathbb{R}$  and that  $f'g - fg' = 3$  on  $\mathbb{R} - \{0\}$ . It follows that

$$(2) \quad K' = \frac{1}{2}(f'g + fg' + f'g - fg') = f'g \text{ on } \mathbb{R} - \{0\}.$$

Set  $b = f'$  on  $I$ ,  $b^* = f'$  on  $I - \{0\}$ ,  $b^*(0) = \frac{3}{2}$ . There is a function  $v$  such that  $v' = (1+g)^{-1}$  on  $I$ ,  $v(0) = 0$ ;  $v$  is twice differentiable on  $I$ .

Since  $v' > 0$  on  $I$ , we may define a function  $a$  on  $I$  by  $v'' + av' + bv = 0$ . Since  $v(0) = 0$ , we have also

$$(3) \quad v'' + av' + b^*v = 0 \text{ on } I.$$

Further (see (2))  $b^*/v' = b^*(1+g) = f' + f'g = (f+K)'$  on  $I - \{0\}$ ; since  $f'(0) = 0$ , we have  $(b^*/v')(0) = \frac{3}{2} = (f+K)'(0)$ . Thus  $b^*/v' = (f+K)'$  on  $I$ .

According to 19 and (3), (1\*) is normal; by 16,  $a$  has a primitive on  $I$ . As  $(b/v')(0) \neq (b^*/v')(0)$ , the function  $b/v'$  does not have a primitive on  $I$ . By 19, (1) is not normal.



21. Let  $F, G, H$  be functions continuous on  $I$ .  
 Let  $F > 0$  on  $I$ ,  $c \in I$  and let  $\alpha, \beta$  be numbers.  
 Then there is a function  $u$  such that  $u(c) = \alpha$ ,  
 $u'(c) = \beta$  and

$$(4) \quad (Fu' + Gu)' = Hu' \quad \text{on } I.$$

Proof. Set  $E = 1/F$ ,  $\gamma = F(c) \cdot \beta + G(c) \cdot \alpha$ ;  
 further let  $u_0$  be the zero function on  $I$ . We  
 define, by induction, functions  $u_1, u_2, \dots$  as follows:  
 If  $u_n$  has a continuous derivative on  $I$ , we set

$$(5) \quad u_{n+1}(x) = \alpha + \int_c^x E(t) (\gamma - G(t)u_n(t) + \int_c^t Hu_n'(t) dt) dt \quad (x \in I).$$

Obviously

$$(6) \quad u_{n+1}'(x) = E(x) (\gamma - G(x)u_n(x) + \int_c^x Hu_n'(t) dt)$$

is continuous on  $I$  ( $n = 0, 1, \dots$ ) and

$$(7) \quad u_n(c) = \alpha \quad (n = 1, 2, \dots).$$

Define  $f_n = u_{n+1}' - u_n'$  ( $n = 1, 2, \dots$ ). Since  $u_{n+1}(c) - u_n(c) = 0$ , we have

$$(8) \quad u_{n+1}(x) - u_n(x) = \int_c^x f_n(t) dt \quad (n = 1, 2, \dots; x \in I).$$

It follows easily from (6) and (8) that

$$(9) \quad f_{n+1}(x) = E(x) \int_c^x (H(t) - G(x)) f_n(t) dt$$

$$(n = 1, 2, \dots; x \in I).$$

Let  $J$  be a compact interval,  $c \in J \subset I$ . There are  
 numbers  $M, N, P$  such that  $|f_1(x)| \leq M$ ,  $|E(x)| \leq N$ ,

$|H(t) - G(x)| \leq P$  for any  $x, t \in J$ . Now it is easy to prove by induction that

$$(10) \quad |f_{n+1}(x)| \leq M(NP|x-c|)^n/n! \quad (x \in J; n = 0, 1, \dots).$$

It follows that  $\langle u'_n \rangle$  converges locally uniformly on  $I$ . By (7),  $\langle u_n \rangle$  converges locally uniformly on  $I$  as well. Let  $u_n \rightarrow u$ . Then  $u'_n \rightarrow u'$ . According to (6),

$$(11) \quad u'(x) = E(x) \left( \gamma - G(x)u(x) + \int_c^x Hu' \right)$$

so that  $(Fu' + Gu)(x) = \gamma + \int_c^x Hu' \quad (x \in I)$ ; this implies

(4). By (7), (11) and the definition of  $\gamma$  we have  $u(c) = \alpha$ ,  $u'(c) = \beta$ .

22. The following three conditions are equivalent to each other:

- i) There are functions  $F, G$  such that  $F > 0$ ,  $a = F'/F$ ,  $b = G'/F$  on  $I$ .
- ii) There are functions  $A, G$  such that  $A' = a$ ,  $G' = b \cdot e^A$  on  $I$ .
- iii) The equation (1) is normal.

Proof. It is obvious that the conditions i) and ii) are equivalent to each other.

1) Let i) hold. Let  $c \in I$ . It follows from 21 that there are functions  $u, v$  such that  $u(c) = 0$ ,  $u'(c) = v(c) = 1$  and that

$$(12) \quad (Fy' + Gy)' = Gy'$$

on  $I$  for  $y = u, v$ . The equation (12) is, obviously, equivalent to

$$(13) \quad (Fy')' + G'y = 0 .$$

If  $y$  satisfies (13), then  $y' = Fy'/F$  is differentiable so that  $y'' + (F'/F)y' + (G'/F)y = 0$ . We see that  $u, v$  solve (1). Since  $a$  has a primitive and  $W_{u,v}(c) = 1$ , the functions  $u, v$  form a fundamental system of (1) (see 10).

2) If (1) is normal, then ii) holds by 16.

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