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On a problem of M. Laczkovich

Let $f(x,y)$ be a function of two real variables. Suppose that the second order partial derivative $f_{xy}(x,y)$ exists at every point. Then the function $g(x,y) = f_x(x,y)$ satisfies the following conditions:

(a) $g(\cdot,y_0)$ is a Baire 1 function for every fixed y_0 ($g(\cdot,y_0)$ is the derivative of $f(\cdot,y_0)$);

(b) $g(x_0,\cdot)$ is continuous for every fixed x_0 (in fact, it is differentiable).

It is easily seen that a function with properties (a) and (b) belongs to the second class of Baire. Consequently, f_{xy} is a Baire 3 function.

M. Laczkovich raised the problem whether f_{xy} is a Baire 1 function. We answer this question in the negative. Our method of construction gives a Baire 2 function; thus, the problem whether f_{xy} is always a Baire 2 function remains open. It should be added that nothing analogous can be said about f, since f_{xy} may be identically zero even for a nonmeasurable function f.

To carry out our construction we need a result of Zahorski (see [1], p. 29, Lemma 12).

Zahorski's Lemma. If H_1 and H_2 are disjoint $\mathcal{G}_\delta$ sets in [0,1] and if they are closed in the Denjoy topology, then there exists a function, a, approximately continuous on [0,1] such that

$$a(x) = 0 \quad (x \in H_1),$$

$$a(x) = 1 \quad (x \in H_2),$$

$$0 < a(x) < 1 \quad (x \notin H_1 \cup H_2)$$

We recall some well known facts about the Denjoy (or density) topology and approximately continuous functions.

(1) The Denjoy open (or briefly D-open) sets are those having (inner) density 1 at each of their points.

(2) These sets are measurable, form the Denjoy topology, and the D-continuous real functions are exactly the approximately continuous functions.

(3) Each approximately continuous function is a Baire 1 function and, if it is bounded, it is a derivative (of its integral).

The reader unfamiliar with these concepts and results is referred to [2] or [3]. Without loss of generality we may confine our construction to the unit square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

Theorem. There is a function $f(x,y)$ such that $f_{xy}(x,y)$ exists at every point of the square $0 \leqq x \leqq 1$,

$0 \leqq y \leqq 1$ but $f_{xy}(x,y)$ is not a Baire 1 function.

Proof. Let a continuous function $\varphi(x)$ $(0 \leqq x \leqq 1)$ satisfying $0 \leqq \varphi(x) \leqq 1$ be given in advance. Let $s_n(x,y) = 2^{-n} \cdot \sin[n2^n \cdot (y - \varphi(x))]$ $(n = 1,2,\ldots)$.

We need the following properties of s_n:

(4) $|s_n| \leqq 2^{-n}$;

(5) $\left|\frac{\partial s_n}{\partial y}\right| \leqq n$;

(6) $\frac{\partial s_n}{\partial y}(x, \varphi(x)) = n$;

(7) s_n is continuous.

Let $r_1, r_2, \ldots$ be an enumeration of the rational numbers of $[0,1]$, let G_0 be the empty set and let a_0 be the zero function on $[0,1]$. We are going to construct by induction numbers ζ_k, sets G_k and functions a_k with the following properties:

(i) $\zeta_k \in [0,1]$, $|\zeta_k - r_k| < 2^{-k}$;

(ii) $G_k \supset G_{k-1}$, G_k is a $\mathcal{G}_\delta$ and D-closed set in $[0,1]$ such that both G_k and $[0,1] \setminus G_k$ are dense in $[0,1]$;

(iii) a_k is an approximately continuous function on $[0,1]$ such that

$$a_k(\zeta_k) = 1,$$

$$a_k(x) = 0 \quad (x \in G_k),$$

$$0 < a_k(x) \leqq 1 \text{ and } a_{k-1}(x) < 2^{-k+1} \quad (x \notin G_k)$$

$$(k = 1,2,\ldots) \quad .$$

Put $\zeta_1 = r_1$ and let $G_1 \subset [0,1]\setminus\{r_1\}$ be an arbitrary $\mathcal{G}_\delta$ set of measure 0, dense in $[0,1]$. By Zahorski's Lemma we can find a_1 fulfilling (iii) with $k = 1$. It is obvious that conditions (i) and (ii) are fulfilled for $k = 1$ as well.

Suppose that n is a natural number and that we have already constructed ζ_k, G_k and a_k $(k \leqq n)$ such that the conditions (i) - (iii) hold for $k = 1,\ldots,n$. Define $A_n = \{x; a_n(x) \geqq 2^{-n}\}$. According to (2) and (3), A_n is a $\mathcal{G}_\delta$ and D-closed set. Obviously $A_n \cap \bar{G}_n = \emptyset$. Set $G_{n+1} = A_n \cup G_n$. It follows easily from (ii) with $k = n$ and from the Baire category theorem that $[0,1]\setminus G_{n+1}$ is dense. We see that (ii) holds with $k = n + 1$. In particular, we can select a number $\zeta_{n+1} \in [0,1]\setminus G_{n+1}$ with $|\zeta_{n+1} - r_{n+1}| < 2^{-n-1}$. Now we apply Zahorski's lemma to the pair of sets G_{n+1}, $\{\zeta_{n+1}\}$ and we obtain a function a_{n+1} such that (iii) holds with $k = n + 1$. This completes our construction.

Define

(9) $$g(x,y) = \sum_{n=1}^{\infty} a_n(x) s_n(x,y) \quad .$$

Since $0 \leqq a_n \leqq 1$ and $|s_n| \leqq 2^{-n}$, the series on the right hand side is uniformly convergent. It follows that $g(\cdot, y_0)$ is approximately continuous and bounded for any fixed y_0. Putting

(10) $$f(x,y) = \int_0^x g(t,y)\,dt$$

we obtain a function with $f_x = g$.

Next we prove that g_y exists. Fix an $x_0 \in [0,1]$. If $x_0 \notin \bigcup_{n=1}^{\infty} G_n$, then, by (iii), $a_n(x_0) < 2^{-n}$ for each n. Thus, by (5), the series

$$\sum_{n=1}^{\infty} a_n(x_0) \frac{\partial s_n}{\partial y} (x_0, \cdot)$$

is uniformly convergent. Hence,

$$g_y(x_0, y) = \sum_{n=1}^{\infty} a_n(x_0) \frac{\partial s_n}{\partial y} (x_0, y) \quad . \tag{11}$$

If, however, there is an index N such that $x_0 \in G_N$, then (see (ii) and (iii)) $a_n(x_0) = 0$ for each $n \geqq N$ and (11) holds again. We have, in any case,

$$f_{xy}(x,y) = g_y(x,y) = \sum_{n=1}^{\infty} n a_n(x) \cos[n 2^n \cdot (y - \varphi(x))] \; . \tag{12}$$

Let S be any open disc in the unit square and let $M > 1$. Since, by (i), the sequence $\{\zeta_n\}$ is dense in $[0,1]$, we can find integers ν, k and j such that

$$2^{\nu} - \nu^2 > M, \quad k > 2^{\nu}, \quad \text{and} \tag{13}$$

$$(\zeta_k, \varphi(\zeta_k) + j \frac{2\pi}{2^{\nu}}) \in S \quad .$$

(Obviously $\nu > 0$, $-2^{\nu} < j < 2^{\nu}$.) Set $y = \varphi(\zeta_k) + j \frac{2\pi}{2^{\nu}}$. As

$$\left| \sum_{n=1}^{\nu-1} n a_n(\zeta_k) \cos(nj \frac{2\pi}{2^{\nu}} 2^n) \right| \leqq \sum_{n=1}^{\nu-1} n < \nu^2$$

and

$$\sum_{n=\nu}^{\infty} n a_n(\zeta_k) \cos(nj \frac{2\pi}{2^{\nu}} 2^n) = \sum_{n=\nu}^{\infty} n a_n(\zeta_k)$$

$$\geqq k a_k(\zeta_k) = k > 2^{\nu},$$

we have, by (12), $f_{xy}(\zeta_k, y) > M$ and hence, $\sup_S f_{xy} = +\infty$. This shows that f_{xy} does not belong to the first class of Baire.

We remark that

(iv) $f_{xy}(x,y) = 0$ on the dense set $G_1 \times [0,1]$ (This follows immediately from (12), (ii) and (iii).);

(v) the function φ plays no particular role in the proof (we could take $\varphi \equiv 0$). It only gives a little flexibility in locating the points where f_{xy} takes great values.

Problems.

1. May f_{xy} belong to the third but not to the second class of Baire? If both f_{xy} and f_{yx} exist, then, by a theorem of M. Laczkovich, f_x and f_y are Baire 1 functions; thus, f_{xy} and f_{yx} are Baire 2 functions.

2. Is there a function f such that f_y and f_{xy} exist everywhere while f_{yx} does not exist at any point?

3. Suppose that both f_{xy} and f_{yx} exist everywhere. Do they agree at some points? Are they necessarily Baire 1 functions? It is easily seen that f_{xy} and f_{yx} have the same upper and the same lower envelope. Therefore, if one of them is continuous at a given point P, so is the other and $f_{xy}(P) = f_{yx}(P)$. Hence, if f_{xy} and f_{yx} are Baire 1 functions, they agree on a dense $\mathcal{G}_\delta$ set.

References

[1] Z. Zahorski, Sur la première dérivée, Trans. Amer. Math. Soc., 69/1950/, 1-54.

[2] A.M. Bruckner, Differentiation of Real Functions /Lecture Notes in Mathematics, No. 659/, Springer Verlag, 1978.

[3] G. Goffmann, C. Neugebauer, T. Nishiura, Density topology and approximate continuity, Duke Math. J. 28/1961/, 497-506.

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