RESEARCH ARTICLES *Real Analysis Exchange Vol. 5 (1979-80)*

Casper Goffman[1], Department of Mathematics, Purdue University, West Lafayette, Indiana 47907 and Fon-che Liu[2], Academia Sinica, Nankang, Teipei

Lusin Type Theorems for Functions of Bounded Variation

Our purpose is to give a simple proof of the one dimensional version of a deep approximation theorem of J. H. Michael [2]. Let Q be the unit cube in n space. The theorem, in a form given in [1], asserts that if $f:Q \to R$ is of bounded variation in the sense of Cesari then, for each $\varepsilon > 0$, there is a $g:Q \to R$, of class C^1, such that $f(x) = g(x)$ except on a set of measure less than ε, and $|\alpha_f(Q) - \alpha_g(Q)| < \varepsilon$, where $\alpha_h(Q)$ is the area of the surface given by a measurable real function h defined on Q.

Let $I = [0,1]$, and let λ be Lebesgue measure. For each $f:I \to R$, of bounded variation, we consider a measure α_f on the Borel sets in I, called the length measure. To define α_f, first choose the right continuous version of f. (This member of the Lebesgue equivalence class of f minimizes the variation measure which then

[1] This author's contribution supported in part by a grant from the NSF.

[2] This author's contribution made in part by a visit to the Department of Mathematics and Statistics at Purdue University.

agrees with the measure which is the derivative of f in the distribution sense.) Let μ_f be this variation measure. For each Borel set $E \subset I$, let

$$\alpha_f(E) = \sup \sum_{i=1}^{n} \{[\mu_f(E_i)]^2 + [\lambda(E_i)]^2\}^{1/2},$$

where the supremum is taken over all partitions of E into finitely many pairwise disjoint Borel sets. The measure α_f may also be obtained as follows. For each $x \in I$, let J_x be the interval whose end points are the right and left limits of $f(\xi)$ as $\xi \to x$. J_x is a single point except for a countable set of values of x. For each Borel set E, $\alpha_f(E)$ is the one dimensional Hausdorff measure of the planar set $\bigcup_{x \in E} J_x$. For the case where f is absolutely continuous, we also have

$$\alpha_f(E) = \int_E \{1 + [f'(x)]^2\}^{1/2}\, dx.$$

We need a special case of a result of Whitney [3], which we state as a lemma.

<u>Lemma 1.</u> If $f:I \to R$ is differentiable almost everywhere, then for each $\epsilon > 0$ there is a $g:I \to R$, of class C^1, such that $f(x) = g(x)$ except on a set of measure less than ϵ.

We also need the following simple fact which we state without proof.

Lemma 2. Given two points (a,c) and (b,d), with $a < b$, let L be the distance between (a,c) and (b,d), and let α and β be real numbers. For each $\epsilon > 0$, there is an $f:[a,b] \to R$ which is of class C^1, with left derivative at a equal to α and right derivative at b equal to β, such that $\alpha_f([a,b]) < L + \epsilon$.

Theorem 1. A function $f:I \to R$ which is of bounded variation is equivalent to an absolutely continuous function if and only if, for each $\epsilon > 0$, there is a $g \in C^1$ such that, if $G = [x:f(x) \neq g(x)]$ then $\alpha_f(G) < \epsilon$ and $\alpha_g(G) < \epsilon$.

Proof. For the sufficiency, suppose f satisfies the stated condition. Since f is of bounded variation, $\int_I \sqrt{1 + [f'(x)]^2}\, dx < \infty$. Let $\epsilon > 0$, and let $g \in C^1$ be such that if $G = [x:f(x) \neq g(x)]$ then $\alpha_f(G) < \epsilon$ and $\alpha_g(G) < \epsilon$. Now,

$$\alpha_f(I) = \alpha_f(G) + \alpha_f(I\setminus G) = \alpha_f(G) + \alpha_g(I\setminus G)$$
$$\leq \int_{I\setminus G} \sqrt{1 + [f'(x)]^2}\, dx + \epsilon$$

since $f' = g'$ at almost every point of the set where $f = g$. It follows that

$$\alpha_f(I) \leq \int_I \{1 + [f'(x)]^2\}^{1/2}\, dx$$

so that f is absolutely continuous.

For the converse, suppose f is absolutely continuous. Let $\epsilon > 0$. There is a $\delta > 0$ such that for every Borel set E, with $\lambda(E) < \delta$, we have

$$\alpha_f(E) = \int_E \{1 + [f'(x)]^2\}^{1/2}\, dx < \epsilon/3.$$

By Lemma 1, there is a function $h \in C^1$ such that $f(x) = h(x)$, except on a set E with $\lambda(E) < \delta$. We take E, as we may by adding a countable set, so that $A = I\setminus E$ is perfect. Then E is the union of pairwise disjoint, nonabutting open intervals $I_1, I_2, \ldots$. Since $h \in C^1$, the series $\sum_{n=1}^{\infty} \alpha_h(I_n)$ converges. Choose m so that

$\sum_{n=m+1}^{\infty} \alpha_h(I_n) < \epsilon/3$. Now $\alpha_f(E) < \epsilon/3$, but $\alpha_h(E)$ may be too big. We modify h on the finite set of intervals $I_1, \ldots, I_m$ to obtain an appropriate g of class C^1. By Lemma 2, we may define g on these intervals so that

$$\sum_{n=1}^{m} \alpha_g(I_n) < \sum_{n=1}^{m} \alpha_f(I_n) + \epsilon/3 \le \alpha_f(E) + \epsilon/3 .$$

Then

$$\alpha_g(E) = \sum_{n=m+1}^{\infty} \alpha_h(I_n) + \sum_{n=1}^{m} \alpha_g(I_n) < \epsilon/3 + \alpha_f(E) + \epsilon/3 \leq \epsilon.$$

Theorem 2. If $f:I \to R$ is of bounded variation then, for each $\epsilon > 0$, there is a $g:I \to R$ of class C^1 such that $f(x) = g(x)$ except on a set of measure less than ϵ and $|\alpha_f(I) - \alpha_g(I)| < \epsilon$.

Proof. Suppose $f:I \to R$ is of bounded variation and right continuous. By Lemma 1, there is a function $u \in C^1$ such that $f(x) = u(x)$ except for a set E of Lebesgue measure less than ϵ. We choose E, as we may, so that $A = I\setminus E$ is perfect and f is continuous at every point of A. The set E is the union of pairwise disjoint nonabutting open intervals $I_1, I_2, \ldots$. As before, there is an m such that $\sum_{n=m+1}^{\infty} \alpha_u(I_n) < \epsilon/2$.

By Lemma 2, we may modify u on the intervals $I_1, \ldots, I_m$, to obtain $v \in C^1$ such that $\sum_{n=1}^{m} \alpha_v(I_n) < \sum_{n=1}^{m} \alpha_f(I_n) + \epsilon/2$.

Now, $f(x) = v(x)$, except on a set E of measure less than ϵ and $\alpha_v(I) < \alpha_f(I) + \epsilon$. Finally, we modify v on I_1 to obtain a longer $g \in C^1$, but only long enough to have

$$\alpha_f(I) - \epsilon < \alpha_g(I) < \alpha_f(I) + \epsilon$$

References

1. C. Goffman, Approximation of nonparametric surfaces of finite area, J. Math. Mech. 12 (1963), 737-745.

2. J. H. Michael, The equivalence of two areas for nonparametric discontinuous surfaces, Illinois J. of Math. 7 (1963), 59-78.

3. H. Whitney, On totally differentiable and smooth functions, Pacific J. Math. 1, (1951), 143-159.

Received October 10, 1979