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A generalization of the total differential and of the Lipschitz condition.

Slobodnik in [3] has proved that if f fulfils locally the approximate Lipschitz condition (that is, there exists a number K and, for every x_0, there exists a measurable set A_{x_0} such that its density at x_0 is equal to 1 and, for every $x \in A_{x_0}$, $|f(x) - f(x_0)| \leq K \,|\, x - x_0|$), then f fulfils the ordinary Lipschitz condition. Replacing " density equal to 1 " by some weaker condition such as " unilateral lower densities greater that $\frac{1}{2}$ " we shall give several similar results for the real function of one or two real variables.

Let m(A) and $m_2(A)$ denote the Lebesgue measure for the set A on the line and on the plane, respectively.

Definition 1.

Let f be a real function defined on the real line. If there exists a number $K > 0$ and, for every $x_0 \in R$, there exists a measurable set E_{x_0} such that

1) $$\liminf_{h \to 0^+} \frac{m(E_{x_0} \cap [x_0, x_0 + h])}{h} > \frac{1}{2}$$

and

$$\liminf_{h\to 0^+} \frac{m(E_{x_0} \cap [x_0 - h, x_0])}{h} > \frac{1}{2}$$

2) for every $x \in E_{x_0}$, $|f(x) - f(x_0)| \le K|x - x_0|$,

then f is said to fulfill the strongly preponderant approximate Lipschitz condition.

Definition 2.

Let f be a real function defined on R^2.

If there exists a number $K > 0$ and, for every $(x_0,y_0) \in R^2$, there exists a measurable set $E_{(x_0,y_0)}$ such that

1) $$\liminf_{h\to 0^+} \frac{m_2(E_{(x_0,y_0)} \cap [x_0,x_0+h] \times [y_0,y_0+h])}{h^2} > \frac{1}{2},$$

$$\liminf_{h\to 0^+} \frac{m_2(E_{(x_0,y_0)} \cap [x_0,x_0+h] \times [y_0-h,y_0])}{h^2} > \frac{1}{2},$$

$$\liminf_{h\to 0^+} \frac{m_2(E_{(x_0,y_0)} \cap [x_0-h,x_0] \times [y_0,y_0+h])}{h^2} > \frac{1}{2},$$

and

$$\liminf_{h\to 0^+} \frac{m_2(E_{(x_0,y_0)} \cap [x_0-h,x_0] \times [y_0-h,y_0])}{h^2} > \frac{1}{2},$$

2) for every $(x,y) \in E_{(x_0,y_0)}$,

$$|f(x,y) - f(x_0,y_0)| \le K \, \| (x,y) - (x_0,y_0) \| ,$$

then f is said to fulfill the strongly preponderant approximate Lipschitz condition.

At present, we shall formulate two theorems giving the relationship between the strongly preponderant approximate Lipschitz condition and the ordinary Lipschitz condition.

Theorem 1.

If the real function f defined on the line fulfills the strongly preponderant approximate Lipschitz condition with the number K, then f fulfills the ordinary Lipschitz condition with the same number.

Theorem 2.

If the real function f defined on R^2 fulfills the strongly preponderant approximate Lipschitz condition with the number K, then f fulfills the ordinary Lipschitz condition with the number 2K.

The function $f(x,y) = y$ defined on R^2 shows that the number K in Theorem 2 cannot be preserved because, for every $(x_0,y_0) \varepsilon R^2$, there exists a set

$E(x_0,y_0) = \{(x,y) \varepsilon R^2 :. \ |x-x_0| \leq L|y-y_0| \ L > 1\}$ such that, for every $(x,y) \varepsilon E(x_0,y_0)$,

$$|f(x,y) - f(x_0,y_0)| \leq \frac{L}{\sqrt{1 + L^2}} \| (x,y) - (x_0,y_0) \| .$$

The number $\frac{L}{\sqrt{1 + L^2}}$ is less than 1 and the function $f(x,y) = y$ fulfills the ordinary Lipschitz condition with the number 1. Considering certain regular sets $E(x_0,y_0)$ in definition 2, we can formulate the preponderant approximate Lipschitz condition which assures the preservation of the number K in the ordinary Lipschitz condition. It is the ψ-regular approximate Lipschitz condition.

Definition 3.

Let $\psi: R^+ \to R$ be a strongly non-decreasing function for which there exists a number $C > 0$ such that, for every $x_1, x_2 \epsilon R^+$,

$$\frac{1}{C} |x_2 - x_1| \leq |\psi(x_2) - \psi(x_1)| \leq C|x_2 - x_1|.$$

Let $(x_0, y_0) \epsilon R^2$. Denote by $\Phi(x_0, y_0)$ the following family of rectangles:

1) (x_0, y_0) is the point of intersection of diagonals of all rectangles in $\Phi(x_0, y_0)$

2) the right upper vertex of every rectangle lies on the graph of the function $y = \psi(x - x_0) + y_0$

3) edges of the rectangles are parallel to the coordinate axes. We shall call such a family of rectangles a ψ-regular family of rectangles.

If the lower density of the set $\bigcup_{C \epsilon \Phi(x_0, y_0)} \text{fr } C$ (where fr C denotes the edge C) is greater than $\frac{1}{2}$, then $\Phi(x_0, y_0)$ is said to be a thick ψ-regular family of rectangles.

If the density of the set $\bigcup_{C \epsilon \Phi(x_0, y_0)} \text{fr } C$ is equal to 1, then $\Phi(x_0, y_0)$ is said to be a very thick ψ-regular family of rectangles.

Definition 4.

Let f be a real function defined on R^2. If there

exists a number $K > 0$ and, for every $(x_0,y_0) \varepsilon R^2$, there exists a thick ψ-regular family of rectangles $\Phi(x_0,y_0)$ such that, for $(x,y) \varepsilon \bigcup_{C \varepsilon \Phi(x_0,y_0)} \text{fr } C$,
$|f(x,y) - f(x_0,y_0)| \leq K \| (x,y) - (x_0,y_0) \|$, then f is said to fulfill the ψ-regular approximate Lipschitz condition.

Theorem 3

Let f be a real function defined on R^2. If f fulfills the ψ-regular approximate Lipschitz condition with the number K, then f fulfills the ordinary Lipschitz condition with the same number.

On the ground of the definition of the ψ-regular family of rectangles, we shall introduce a definition of a ψ-regular approximate differential. It is a generalization of the regular approximate differential which was discussed by Fadell in [1]. Fadell has proved a theorem similar to Theorem 4 for the function $\psi(x) = x$.

Definition 5.

Let f be a real function defined on $S \subset R^2$. We shall say that f has a ψ-regular approximate differential at $(x_0,y_0) \varepsilon S$ if and only if there exists a very thick ψ-regular family of rectangles $\Phi(x_0,y_0)$ such that $f| \bigcup_{C \varepsilon \Phi(x_0,y_0)} \text{fr } C \cap S$ has a total differential at (x_0,y_0).

Theorem 4.

Let f be a continuous real function defined on the open bounded subset $S \subset R^2$. If we assume that f has first partial derivatives almost everywhere on S, then f has the ψ-regular approximate differential almost everywhere on S.

The last-mentioned theorem is based on the following lemma.

Lemma.

If $(x_0, y_0) \in R^2$ is a point of the linear density of the measurable set $S \subset R^2$ in the directions of the coordinate axes, then there exists a very thick ψ-regular family of rectangles $\Phi(x_0, y_0)$ such that the lines passing through (x_0, y_0), parallel to the coordinate axes, intersect the edges of every rectangle at points of the set S.

The following theorem is a generalization of the theorem proved by Slobodnik in [3].

Theorem 5.

Let f be a real function defined on the open convex subset $S \subset R^2$. If f has a ψ-regular approximate differential at every point of S and if there exists a number $K > 0$ such that

$$\sup_{\rho((x,y),(x', y')) \le 1} |\nabla f_{\psi\text{-ap}}(x,y) \circ ((x,y) - (x',y',))| \le K$$

for every $(x,y) \in S$ (where $\nabla f(x,y)$ denotes a gradient of f and $\rho((x,y),(x',y',))$ denotes a distance between (x,y) and (x',y') , then f is differentiable on S and $\nabla f(x,y) = \nabla f_{\psi\text{-ap}}(x,y)$ and, for every $(x,y),(x',y') \in S$,

$$|f(x,y) - f(x',y')| \leq K \, \| (x,y) - (x',y') \| .$$

Theorem 6.

If the real function f defined on the rectangle $P \subset R^2$ fulfills the ψ-regular approximate Lipschitz condition, then f has a total differential almost everywhere on P.

References

[1] A. Fadell, On the existence of regular approximate differentials. Proc. AMS 37, Number 2, 1973, 541-544.

[2] E. Lazarow, Doctoral thesis, University of Lodz, Poland, 1979.

[3] S.G. Slobodnik, On approximately differentiable functions of several variables, Mat. Zametki, 17 1975, 857-871 (in Russian).

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