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CHARACTERIZING CLUSTER SETS OF REAL FUNCTIONS

Section 1.

If f is any real function defined on a real interval I, the left and right cluster sets of f at x may be defined by the sets

$$\{y\colon \text{there is a sequence } \{x_n\}_{n=1}^{\infty} \text{ so that } x_n \uparrow x \text{ and } f(x_n) \to y\}$$

and

$$\{y\colon \text{there is a sequence } \{x_n\}_{n=1}^{\infty} \text{ so that } x_n \downarrow x \text{ and } f(x_n) \to y\}$$

respectively.

We will denote these sets by $C^-f(x)$ and $C^+f(x)$ respectively.

By augmenting the cluster set by the functional value at each point, we obtain the sets

$$K^-f(x) = \{f(x)\} \cup C^-f(x)$$

and

$$K^+f(x) = \{f(x)\} \cup C^+f(x).$$

Thus for each real function f, we obtain the above four set-valued maps.

By abstracting the properties of these set-valued maps, we obtain the concept of a cluster set structure (see the precise definition below). Then it is possible to characterize certain classes of functions in terms of a cluster set structure in the sense that each such function gives rise to a cluster set structure and a given cluster set structure can be generated by such a function.

Ceder, in [2], gave characterizations of the families of cluster sets corresponding to arbitrary functions, Baire 2 functions, Darboux functions, and Darboux Baire 2 functions. We defer the explicit statement of theorems until the necessary terminology is introduced.

Let E be the real line. Unless otherwise stated, all functions will be real valued and defined on an interval I. We make no distinction between a function and its graph.

Let L and R be set-valued maps on I into the nonvoid closed subsets of E. We put

$$L^*(x) = \bigcap_{n=1}^{\infty} \{cl(\bigcup \{L(y) \cup R(y): x - 1/n < y < x\})\}$$

and

$$R^*(x) = \bigcap_{n=1}^{\infty} \{cl(\bigcup \{L(y) \cup R(y): x < y < x + 1/n\})\}.$$

The pair (L,R) is called a <u>cluster set structure</u> (css) provided for all x we have

(1) $L(x) \cap R(x) \neq \emptyset$.

(2) $L^*(x) \subset L(x)$ and $R^*(x) \subset R(x)$.

(3) Letting $r(x) = R(x) - R^*(x)$ and $l(x) = L(x) - L^*(x)$ then $\operatorname{card}(r(x)) \leq 1$ and $\operatorname{card}(l(x)) \leq 1$.

(4) $r(x) \neq \emptyset$ and $l(x) \neq \emptyset$ imply $r(x) = l(x)$.

(5) $r(x) - L^*(x) \neq \emptyset$ implies $l(x) \neq \emptyset$ and $l(x) - R^*(x) \neq \emptyset$ implies $r(x) \neq \emptyset$.

A cluster set structure (L,R) is called a <u>connected cluster set structure</u> (ccss) provided $L(x)$ and $R(x)$ are connected sets, $L(x) = L^*(x)$, and $R(x) = R^*(x)$ for all x. We define graph (L,R) to be the planar set $\cup \{\{x\} \times (L(x) \cup R(x)): x \in I\}$.

We now state the major results of Ceder in [2].

<u>Theorem 1</u> (Ceder). For any function f, (K^-f, K^+f) is a css.

<u>Theorem 2</u> (Ceder). If (L,R) is a css, then there is a Baire 2 function f such that $K^-f = L$ and $K^+f = R$.

<u>Theorem 3</u> (Ceder). If f is a Darboux function, then (K^-f, K^+f) is a ccss.

<u>Theorem 4</u> (Ceder). If (L,R) is a ccss, then there is a Darboux Baire 2 function f such that $K^-f = L$ and $K^+f = R$.

Section 2.

Suppose that (L,R) is a css on I. Let $G = \text{graph}(L,R)$ and assume that $\text{rng}(G)$ is bounded. Then the functions defined by

$$u(x) = \sup(L(x) \cup R(x)) \quad \text{and} \quad \ell(x) = \inf(L(x) \cup R(x))$$

are easily seen to be upper semicontinuous and lower semicontinuous respectively. If $\text{rng}(G)$ is unbounded, we put $u(x) = \infty$ or $\ell(x) = -\infty$ if $L(x) \cup R(x)$ is unbounded above or below respectively.

Let E' denote the extended real numbers. We consider all non-void subsets of $I \times E'$ of the form $J \times H$ where J and H are open connected subsets of I and E' respectively. We refer to such sets as open rectangles.

Let $\mathcal{U}(x)$ be the collection of all open rectangles containing the point $(x,u(x))$ and missing ℓ. Similarly, let $\mathcal{L}(x)$ be the collection of all open rectangles containing the point $(x,\ell(x))$ and missing u.

A css (L,R) is said to satisfy condition D if

(1) for every open rectangle S hitting G, we have $S \cap u \neq \emptyset$ or $S \cap \ell \neq \emptyset$;

(2) there do not exist two non-void sets J and K such that

(i) $u(x) - \ell(x) \geq \delta > 0$ for all $x \in J \cup K$ and some $\delta > 0$,

(ii) $cl(J) = cl(K) = P$ a perfect set,

(iii) for all $x \in J$ (resp. K) there is an $S \in \mathcal{U}(x)$ (resp. $\mathcal{L}(x)$) such that $dom(S \cap G) \subset P$.

Ceder, in [2], posed the problem of characterizing the cluster sets of Darboux Baire 1 functions. In [3], and also in [4], we prove the two following theorems giving such a characterization.

Theorem 2.1 Let f be a Darboux Baire 1 function on I. Then (K^-f, K^+f) is a ccss satisfying condition D.

Theorem 2.2 Let (L,R) be a ccss on I satisfying condition D. Then there is a Darboux Baire 1 function f such that $K^-f = L$ and $K^+f = R$.

Section 3.

If we consider topologies on I other than the Euclidean topology, we may investigate the notions of cluster sets and cluster set structures with respect to these new topologies. The topologies we will be concerned with arise in the following manner.

A σ-ideal W is a collection of subsets of I with the properties

(i) subsets of members of W are members of W,

(ii) W is closed under countable unions.

We consider σ-ideals which, in addition, contain all singleton subsets of I but contain no nondegenerate subintervals of I.

For any such σ-ideal W we may define a topology $\mathfrak{T}(W)$ on I by picking a base to be all sets of the form $(c,d)-A$ where (c,d) is an open subinterval of I and $A \in W$.

For any function f on I, we define the left and right cluster sets of f with respect to $\mathfrak{T}(W)$ by

$$WC^-f(x) = \cap \{C^-f_A(x) \colon A \in W\}$$

and

$$WC^+f(x) = \cap \{C^+f_A(x) \colon A \in W\}$$

respectively, where f_A is f restricted to $I-A$.

In addition we let

$$WK^-f(x) = WC^-f(x) \cup \{f(x)\}$$

and

$$WK^+f(x) = WC^+f(x) \cup \{f(x)\}.$$

By following the definition of a cluster set structure on I with the Euclidean topology, we define a cluster set structure on I with the $\mathfrak{T}(W)$ topology in the following manner. Let WL and WR be set-valued maps on I into the non-void closed subsets of E. We define

$$WL^*(x)$$
$$= \cap \left(\bigcap_{n=1}^{\infty} \{cl(\cup\{WL(y) \cup WR(y) \colon y \in (x - 1/n, x) - A\})\} \colon A \in W\right)$$

and

$WR^*(x)$

$$= \cap \left(\cap_{n=1}^{\infty} \{cl(\cup \{WL(y) \cup WR(y) : y \in (x, x+1/n) - A\} : A \in W \right).$$

The pair (WL, WR) is called a <u>W-cluster set structure</u> (W-css) on I provided the five conditions on a css, replacing L, R, L^* and R^* by WL, WR, WL^* and WR^* respectively, are satisfied for all x.

A W-cluster set structure (WL, WR) is called a <u>W-connected cluster set structure</u> (W-ccss) if $WL(x)$ and $WR(x)$ are connected sets, $WL(x) = WL^*(x)$, and $WR(x) = WR^*(x)$ for all $x \in I$.

It will be useful to define two classes of σ-ideals for future reference.

<u>Definition 3.1</u> A σ-ideal W is said to be of <u>type C</u> if for each sequence of closed sets $\{B_n\}_{n=1}^{\infty}$, each $B_n \notin W$, there exist disjoint closed sets $\{A_n\}_{n=1}^{\infty}$, with $A_n \subset B_n$, and $A_n \notin W$ for all n. If we drop the requirement that each A_n be closed, then W is of <u>type C′</u>.

For example, it is easy to show that the σ-ideal of countable sets and the σ-ideal of Lebesgue measure zero sets are of type C. However, the σ-ideal of first category sets is of type C′ but not C. An unsolved problem is whether every σ-ideal which contains all singletons but no intervals is of type C′.

<u>Definition 3.2</u> For any σ-ideal W and $n \geq 1$, a function f is called WB_n if for every open set V, we have $f^{-1}(V) = (B - A_1) \cup A_2$ where B is of additive Borel class n

and A_1 and A_2 are members of W.

In [3] we prove the following.

Theorem 3.1 If W is a σ-ideal and f is any function mapping I into E, then (WK^-f, WK^+f) is a W-css.

Theorem 3.2 If W is a σ-ideal of type C and (WL, WR) a W-css, then there is a WB_2 function f such that $WK^-f = WL$ and $WK^+f = WR$.

Corollary 3.1 If W is a σ-ideal of type C' and (WL, WR) a W-css, then there is a function f such that $WK^-f = WL$ and $WK^+f = WR$.

One of the generalizations of the Darboux property cited by Bruckner and Ceder in [1] is that of $\langle M,N\rangle$-Darboux. If M and N are families of subsets of E, then f is $\langle M,N\rangle$-Darboux if for each interval $[a,b]$ contained in the domain of f and $A \in M$, there is a $B \in N$ such that

$$[f(a),f(b)] - B \subset f([a,b] - A).$$

It turns out that this is the notion we need to discuss W-connected cluster set structures, as the next two theorems show.

Theorem 3.3 Let W be a σ-ideal and $N = \{B: B$ has empty interior$\}$. For any $\langle W,N\rangle$-Darboux function f, (WK^-f, WK^+f) is a W-ccss.

Theorem 3.4 If W is a σ-ideal of type C and (WL, WR) a W-ccss, then there is a $\langle W,N\rangle$-Darboux function f such that $WK^-f = WL$ and $WK^+f = WR$, where $N = \{B: B$ has empty interior$\}$. In fact, f can be chosen to be Darboux Baire 2.

Corollary 3.2 Let W be a σ-ideal of type C′ such that every set not a member of W has cardinality c. If (WL, WR) is a W-ccss, then there is a $\langle W,N\rangle$-Darboux function f such that $WK^-f = WL$ and $WK^+f = WR$. In fact, f may be chosen to be Darboux.

Theorem 3.5 Let W be a σ-ideal and f a Baire 1 $\langle W,N\rangle$-Darboux (and thus Darboux) function where $N = \{B: B$ has empty interior$\}$. Then (WK^-f, WK^+f) is a W-ccss satisfying condition D of Theorem 2.1.

Theorem 3.6 Let W be a σ-ideal and (WL,WR) a W-ccss which satisfies condition D of Theorem 2.1. Then there is a Baire 1 $\langle W,N\rangle$-Darboux (and thus Darboux) function f such that $WK^-f = WL$ and $WK^+f = WR$, where $N = \{B: B$ has empty interior$\}$.

Section 4.

We also consider, in [3], the notion of a cluster set structure defined on an interval I with the density topology.

We will need the following notation. For any set $A \subset I$ we let $m^*(A)$ be the outer Lebesgue measure of A, and $m(A)$ the Lebesgue measure of a measurable set A. By $d^*(x,A)$ we mean the upper outer density of A at the point x, and by $d(x,A)$ the density of A at x, if it exists. Put $D(x) = \{A: d^*(x,A) = 0\}$.

For any function f on I, we form the left and right cluster sets of f with respect to the density topology $\mathcal{T}(d)$ as follows.

Let

$$dC^-f(x) = \cap\{C^-f_A(x): A \in D(x)\}$$

and

$$dC^+f(x) = \cap\{C^+f_A(x): A \in D(x)\}$$

where f_A is f restricted to $I-A$.

Observe that the cluster sets above cannot be defined in terms of a σ-ideal of sets as in Section 3.

Put

$$dK^-f(x) = dC^-f(x) \cup \{f(x)\}$$

and

$$dK^+f(x) = dC^+f(x) \cup \{f(x)\}.$$

Let dL and dR be set valued maps on I into the family of non-void closed subsets of E. We put

$$dL^*(x) = \cap(\cap_{n=1}^{\infty}\{cl(\cup\{dL(y) \cup dR(y): y \in (x-1/n,x)-A\})\}: A \in D(x))$$

and

$$dR^*(x) = \cap(\cap_{n=1}^{\infty}\{cl(\cup\{dL(y) \cup dR(y): y \in (x,x+1/n)-A\})\}: A \in D(x)).$$

The pair (dL, dR) is called a <u>density cluster set structure</u>

(abbreviated d-css) on I provided for all x we have the five conditions on a css satisfied, replacing L, R, L^*, and R^* by dL, dR, dL^*, and dR^* respectively.

The next two theorems show that our definition of a d-css characterizes the cluster sets of functions defined on an interval I with the density topology.

<u>Theorem 4.1</u> For any function f, (dK^-f, dK^+f) is a d-css.

<u>Theorem 4.2</u> If (dL, dR) is a d-css, then there is a function f such that $dK^-f = dL$ and $dK^+f = dR$.

In Theorem 4.2 we could not say anything about the measurability or Baire classification of the function f because of the type of construction used. However, we do handle the measurable case in the next two theorems.

<u>Theorem 4.3</u> If f is measurable, then $dK^-f(x) = dK^+f(x) = dC^-f(x) = dC^+f(x) = \{f(x)\}$ almost everywhere.

<u>Theorem 4.4</u> Let (dL, dR) be a d-css such that $dL(x) = dR(x) = dL^*(x) = dR^*(x) = \{z(x)\}$, a singleton, on a set of full measure A. Then there is a measurable function f such that $dL = dK^-f$ and $dR = dK^+f$.

We remark that the general problem of Baire classification remains unsolved.

Section 5

For any function f and σ-ideal W it is clear that $WK^-f(x) \subset K^-f(x)$ and $WK^+f(x) \subset K^+f(x)$ for all x. This leads us to ask if similar containment properties on a css (L,R) and a W-css (WL, WR) assure the existence of a function which in some sense realizes both the css and the W-css simultaneously. If we assume that $WL(x) \subset L(x)$ and $WR(x) \subset R(x)$ for all x, it is clear we cannot expect to always construct a function f such that $WK^-f = WL$, $WK^+f = WR$, $K^-f = L$, and $K^+f = R$. However, we can try to construct f so that $WC^-f = WL^*$, $WC^+f = WR^*$, $K^-f = L$, and $K^+f = R$. This then is the sense of realizing cluster values we are interested in. We adopt the notation $(WL, WR) \subset (L,R)$ to mean $WL(x) \subset L(x)$ and $WR(x) \subset R(x)$ for all x.

Now consider two σ-ideals W and W', with $W' \subset W$. Then for any function f, $WK^-f(x) \subset W'K^-f(x)$ and $WK^+f(x) \subset W'K^+f(x)$ for all x. We denote this containment by $(WK^-f, WK^+f) \subset (W'K^-f, W'K^+f)$. The question arises then, given a W-css (WL, WR) and a W'-css $(W'L, W'R)$ with $(WL, WR) \subset (W'L, W'R)$, is there a function f which realizes both simultaneously. To deal with this question we make the following definition.

<u>Definition 5.1</u> Let W and W' be σ-ideals with $W' \subset W$. We say that W <u>covers</u> W' if for every sequence of closed sets $\{B_n\}_{n=1}^{\infty}$ with each $B_n \notin W'$, there exist disjoint sets $A_n \subset B_n$ with $A_n \in W-W'$ for all n.

Theorems 5.1 and 5.2 below are typical of the theorems we prove in [3] concerning two different cluster set structures.

<u>Theorem 5.1</u> Let (L,R) be a css and (WL, WR) a W-css, where W is a σ-ideal of type C, such that $(WL, WR) \subset (L,R)$. Then there is a WB_2 function f such that $WC^-f = WL^*$, $WC^+f = WR^*$, $K^-f = L$, and $K^+f = R$.

<u>Theorem 5.2</u> Let W and W' be σ-ideals of type C and C' respectively, with $W' \subset W$. For any W-css (WL, WR) and W'-css $(W'L, W'R)$ with $(WL, WR) \subset (W'L, W'R)$ there is a WB_2 function f such that $WC^-f = WL^*$, $WC^+f = WR^*$, $W'K^-f = W'L$, and $W'K^+f = W'R$ if and only if W covers W'.

<u>Section 6</u>

For any topological space X and real valued function f defined on X, we consider the cluster set of f at a point $x \in X$ as the set

$$Cf(x) = \{y \in E: (x,y) \in cl(f)\}$$

where the closure of f is simply the closure of f in the space $X \times E$. We put

$$Kf(x) = Cf(x) \cup \{f(x)\}.$$

By considering the essential properties of a cluster set structure defined on an interval of the real line, we can similarly define the notion of a cluster set structure defined on X. We do so in the following manner.

Let H be a set-valued map on X into the non-void closed subsets of E. Put

$$H^*(x) = \cap \{cl(\cup\{H(z): z \in V_\alpha - \{x\}\}) : \alpha \in A\}$$

where $\{V_\alpha: \alpha \in A\}$ is a neighborhood base at x for some index set A. We call (H) a <u>cluster set structure</u> on X if for all x

(1) $H^*(x) \subset H(x)$

(2) letting $h(x) = H(x) - H^*(x)$ then $card(h(x)) \leq 1$.

We investigate which topological spaces X allow a characterization of the cluster sets of real valued functions defined on on X by our definition of a cluster set structure on X. It is clear that for any space X and function f, (Kf) is a cluster set structure (css) on X. We are interested in the converse. We call a space X a <u>cluster space</u> if for every css (H) on X, there is a function f such that $Kf = H$. The following table gives some indication of the nature of cluster spaces. The proofs of statements in the table are given in [3]. An expansion of these results is in progress and will appear later.

We will assume that the space X is dense in itself. If X is discrete, then for any css (H) we would have $H^*(x) = \emptyset$ for all x, implying that $card(H(x)) = 1$ for all x. Thus X is automatically a cluster space, although in a vacuous sense.

I. The space X is a cluster space if X is:

6.1 a complete separable metric space;

6.2 an open subset of a cluster space;

6.3 a closed subset of a cluster space;

6.4 a factor in a product space which is a cluster space;

6.5 a compact Hausdorff space with a base of cardinality less than or equal to c.

II. The space X cannot be a cluster space if X:

6.6 contains a countable open set;

6.7 does not contain countably many disjoint dense sets;

6.8 contains a coutable perfect set.

III. The space X need not be a cluster space if X is:

6.9 a continuous image of a cluster space;

6.10 an F_σ subset of a cluster space;

6.11 given a topology which is finer than one for which X is a cluster space;

6.12 given the sup of two topologies for which X is a cluster space;

6.13 a product of cluster spaces.

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