Real Analysis Exchange Vol. 5 (1979-80)
Paul D. Humke, Deparament of Mathematics, Western Illinois University, Macomb, Illinois 61455

NOWHERE MONOTONE FUNCTIONS AND A PROBLEM OF K. GARG.

1. Introduction. Let X and Y be two topological spaces and f a function mapping X into Y, then for every $y \in Y$, the set $f^{-1}(y) = \{x: f(x)=y\}$ is called a level set (or fiber) of f. The function f is said to be monotone if $f^{-1}(C)$ is connected for every connected subset C of Y (see Kuratowski[9], p. 131), and f is nowhere monotone if f is monotone on no open subset of X, (see [2] and [4]). The function f is said to be connected if f(C) is connected for every connected subset C of X. The study of monotone functions and of nowhere monotone functions has a considerable literature and the interested reader is referred to the bibliography at the end of [4] for a few of the appropriate works. In particular, in [2] and [4] Garg investigated nowhere monotone functions by considering properties of their level sets and in [4] he proves the following result.

Theorem G. Suppose that X is Hausdorff, second countable, and locally connected, and that f is connected and real valued. If f is also nowhere monotone, then there is a residual set of points x in X such that x is a limit point of the level $f^{-1}(f(x))$.

Subsequently he asks ([4] p. 34, Probelm 5.10): if a continuous real valued function f defined on a locally connected, separable, complete metric space X (or on R^n) is nowhere monotone, does there exist a residual set of points x in X such that x is a limit point of the level $f^{-1}(f(x))$ along every simple are in X that has x as an endpoint?

The answer is known to be affirmative if $X=R^1$ (see [1], Theorem 2). In a private communication to Garg, Grande has shown that the completeness hypothesis is a necessary one (see [4] p. 36, Added in proof). The purpose of this note is to answer Garg's question in the negative, and in particular, we prove the following theorem.

Theorem. *There exists a continuous, nowhere monotone real valued function f defined on R^2 such that for every point $x \in R^2$ there is an arc terminating at x along which x is not a limit point of $f^{-1}(f(x))$.*

REFERENCES

[1] K. M. Garg. On nowhere monotone functions. I. Derivates at a residual set, Ann. Univ. Sci. Budapest, Eotvos Sect. Math. 5 (1962), pp. 173-177.

[2] ____ On level sets of a continuous nowhere monotone function, Fund. Math. 52 (1963), pp. 59-68.

[3] ____ Monotonicity, continuity and levels of Darboux functions, Colloq. Math. 28 (1973), pp. 91-103.

[4] ____ Properties of connected functions in terms of their level sets, Fund. Math. 97 (1977) pp. 17-36.

[5] Hocking, J. G. and Young, G. S. Topology. Addison-Wesley, Reading, Mass., 1961.

[6] K. Kuratowski, Topology, Vol. I, New York-London-Warszawa, 1966.

[7] ____ Topology, Vol. II, New York-London-Warszawa, 1968.

[8] J. S. Lipinski, Une remarque sur la continuité et la connexite, Colloq. Math 19 (1968), pp. 251-253.