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Derivative Measures

In one variable, the derivative measure μ of a function f of bounded variation can be any regular finite measure. If f is continuous, then μ is absolutely continuous with respect to Lebesgue measure.

In several variables, bounded variation is replaced by BVC. These are the functions f of n variables such that for each $i = 1,\dots, n$ there is an f^i, equivalent to f, which is of bounded variation in x_i for almost all values of the remaining variables, and the resulting variation function is summable as a function of these $n-1$ variables. If f^i is also coutinuous in x_i for almost all values of the remaining variables, then $f \in \mathcal{L}$, a class which plays the role in several variables which is played by the continuous functions of bounded variation in one variable. If continuity is replaced by absolute continuity in the definition of $\mathcal{L}$, the Sobolev space W_1^1 is obtained.

Consider the partial derivatives $\mu_1,\dots, \mu_n$ of f. The following results are obtained:

a) If $f \in W_1^1$, then μ_i, $i = 1,\dots, n$ is absolutely continuous with respect to Lebesgue n measure.

b) If $f \in BVC$, then μ_i, $i = 1, \ldots n$ is absolutely continuous with respect to Hausdorff n-1 measure.

c) If $f \in \mathcal{L}$ and S is a Borel set with $H^{n-1}(S) < \infty$, then $\mu_i(S) = 0$, $i = 1, \ldots, n$.

This is an extension of the one dimensional facts since $H^0(S)$ is the cardinal number of S.