Real Analysis Exchange Vol. 5 (1979-80)
Robert R. Welland, Department of Mathematics, Northwestern University, Evanston, Illinois 60201

SOME PARTIAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

*Dedicated to Casper Goffman*

1. Introduction. There are several types of first order partial differential equations in Banach spaces for which an existence and uniqueness theory has been developed. We will describe, in this note, some old results and some which have been recently discovered.

Let E, F and G be Banach spaces with norms $\| \|_E$, $\| \|_F$ and $\| \|_G$; let H be Hilbert space with inner product $\langle , \rangle$. Products of spaces will always be assumed to have the product topology and functions will always be assumed to be smooth of order $C^k$ for some $k$ and defined in open balls.

In the early part of this century, Fréchet [2] began the development of infinite dimensional calculus and Dieudonné gave an excellent exposition of this subject in [1]. If $f: E \times \cdots \times F \to G$ is a Fréchet differentiable function then $D_i F_x$ denotes its i-th partial derivative.

2. Cauchy-Kowalewskaya Theorem. M. Zorn [9] developed the theory of infinite dimensional analytic functions, and P. Rosenbloom [7] gave the first proof of the existence and uniqueness, for analytic initial value problems, in this new infinite dimensional setting. He showed that if $f$ and $\phi$ are analytic functions, then there exists a unique local analytic function $u$ which satisfies

$$\text{(i)} \qquad D_1 u_{(t,x)} = f\left(t,x,D_2 u_{(t,x)}\right)$$

and such that $u(0,x) = \phi(x)$ in the domain of $u$. His proof was set up in the realm of complex Banach spaces and used the classical majorant method. An exceedingly direct and simple proof using induction was given recently by Welland and Shinbrot [8].

3. The Liouville Problem. R. Nevanlinna [6] showed that $C^4$ infinite dimension conformal maps from one Hilbert space to itself are compositions of inversions and isometries. His argument was direct and elegant and used very little geometry. Mel Huff [4] developed the relevant concepts in infinite dimensional Riemannian geometry and then used his theory to get Nevanlinna's result for $C^3$-maps [5]. The partial differential equation, in this instance, is non-linear and different from its finite dimensional counterpart in an essential way. The problem is to look for functions $\phi: H \to H$ such that

$$\text{(ii)} \qquad \langle D\phi_x(h), D\phi_x(k)\rangle = \|D\phi_x(h)\| \, \|D\phi_x(k)\| \, \langle h,k\rangle ,$$

is satisfied for all unit vectors $h$ and $k$. Here $D\phi_x(h)$ is the value of the Fréchet derivative $\phi$ at $h$ computed at $x$. The oddness arises in this case because of the presence of the $h$ and $k$ which cannot be eliminated and the fact that (ii) must be satisfied for all choices of $h$ and $k$.

4. The Frobenius Problem. Let $A: E \times F \to L(E,F)$ be $C^1$; here $L(E,F)$ is the Banach space of continuous linear functions from $E$ to $F$. Dieudonné [1] gives an elegant proof of the existence and uniqueness of solutions $u: F \to F$ which satisfy the differential equation

$$\text{(iii)} \qquad Du_x = A\big(x,u(x)\big) .$$

For this result $A$ is required to satisfy an integrability condition and the function $u$ is free to take on any preassigned value at a given point. This result has as a corollary the infinite dimensional version of the Frobenius theorem which says that involutive distributions in infinite dimensional Banach manifolds are locally integrable. There are no dimensional restrictions on the planes of the distribution.

The existence theorem for (iii) together with the definitions and some results of Huff [4] have permitted Welland to show that an infinite dimensional Riemannian manifold with zero curvature is locally isometric to the modelling Hilbert space.

5. Monge Theory. Recently, we have shown that the basic existence and uniqueness theorem for general nonlinear partial differential equations [3] has a counterpart in the infinite dimensional case. The theory takes a slightly different but also slightly more general form.

Let $S = \{x \in H: \|x\| = 1\}$ be the unit sphere in Hilbert space and let $f: H \times S \to R$ be a $C^3$-function. The equation

(iv) $\quad f(x,n) = 0$

takes the place of the usual general partial differential equation which in two dimensions has the form $f(x,y,u,u_x,u_y) = 0$. A local solution of this equation is a submanifold $M$ of codimension 1 whose unit normal $n$ at each point $x$ satisfies (iv). In the classical case this submanifold is simply the graph of the solution. Initial data is replaced by a submanifold $I$ of codimension 2 and a local solution of an initial data problem is a local solution $M$ which intersects $I$ in an open subset. In order to prove the existence and uniqueness, for problems of this sort, it is necessary and sufficient to assume that $D_2f_{(x_0,n_0)} \neq 0$ for some point $(x_0,n_0)$, that $(x_0,n_0) \in (TI_{x_0})^\perp \cap \{x_0,S\}$, that $f(x_0,n_0) = 0$ and that

$$(TI_{x_0})^\perp \cap \{x_0,S\} \quad \text{and} \quad \{(x,n): F(x,n) = 0\}$$

intersect transversally at $(x_0,n_0)$. Here $(TI_{x_0})^\perp$ is the orthogonal complement of the tangent space of $I$ at $x_0$ and $\{x_0,S\}$ is identified with the unit sphere in $H$ about the point $x_0$.

## REFERENCES


[1] Dieudonné, J. Foundations of Modern Analysis, Academic Press, New York (1960).

[2] Fréchet, M. La notion de differentielles dans l'analyse général. Ann. École. Nom. Sup. (3) 42 (1925).

[3] Garabedian, P.R. Partial Differential Equations. John Wiley and Sons, Inc., New York (1964).

[4] Huff, M. Doctoral Dissertation. Northwestern Univ. (1968).

[5] Huff, M. Conformal maps on Hilbert space. Bull. Amer. Math. Soc., (82) 1 (1976).

[6] Nevanlinna, R. On differentiable mappings. Analytic Functions, Princeton Univ. Press, Princeton, N.J. (1960).

[7] Rosenbloom, P. The majorant method. Proc. Symp. Pure Math., Vol. IV, Amer. Math. Soc., Providence, R.I. (1961).

[8] Shinbrot, M. and R. Welland. The Cauchy-Kowalewskaya Theorem. J. Math. Anal. and Applications, 55 (1976).

[9] Zorn, M. Characterization of analytic functions in Banach spaces. Anal. of Math., (2) 46 (1945).