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How Small is a Curve?

We discuss some older and some newer results concerning the extent to which sets (mostly in the plane) that can be regarded as like curves, or rather graphs, must be small in one sense or another.

I. Accessible Sets. A plane set E is accessible if through each of its points passes a line not meeting E again. Banach [1] asked whether a closed accessible set must have measure zero, but Nikodym [11] constructed an accessible F_σ set of full measure in a square, and I constructed one of full measure in R^2 [3]. The plane can't be covered by two accessible sets, but (assuming AC) it can by three, and assuming CH these can be measurable [6]. We don't know whether R^2 is the union of finitely many accessible Borel sets.

Nikodym's construction and mine don't yield only weird counter-examples. Using the same lemmas, I proved [3] that every plane set can be expanded to a union of lines without increasing its measure, and this was what Putnam needed to solve a problem about spectra [12].

II. Disjointly Accessible Sets. Suppose we demand disjoint lines of accessibility. In R^2 that means parallel lines, and we have a rotated graph (see below); a measurable one has measure zero. If we demand only disjoint <u>segments</u> of accessibility (short hairs) then the set, if measurable, must still have measure zero. This fails in R^3, where Larman [9] (solving a problem of Klee and Martin [8]) constructed a compact set of positive measure sprouting disjoint hairs of length 1 (he used my lemmas from [3]). Undoubtedly there's a Borel set of full measure in R^3 sprouting disjoint long hairs, but nobody has constructed it.

III. <u>Rotated Graphs</u>. Assuming AC, it's easy to construct a non-measurable graph of full outer measure: make it meet every non-denumerable compact graph; but a graph still seems small. Mazurkiewicz [10] showed that finitely many rotated graphs can't cover R^2, but Sierpinski [13] deduced from CH that denumerably many can. I showed [4,5] that already AC implies this (however, Sierpinski's graphs were congruent and his only rotation was through $\pi/2$). Indeed, Ceder [2] deduced that they can be connected, and presumably R^2 can be covered by a sequence of disjoint congruent copies of a carefully-constructed connected graph.

Every rectifiable arc has Hausdorff dimension 1, but a graph has dimension 2 if the function oscillates enough. Even if a compact plane set is a graph looked at from every rational direction (and hence from a co-meagre set of directions), it can still have dimension 2: experts see this but might fear the tedious verification, but Fast and I [7] have now discovered how to take the tedium out of such constructions, once and for all.

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