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CLASSICAL PARABOLIC CAPACITY AND QUASILINEAR PARABOLIC EQUATIONS

1. Introduction

The purpose of this note is to describe recent results concerning the behavior of weak solutions of quasilinear parabolic equations of the second order at the boundary of an arbitrary domain. Specifically, we establish a condition for boundary regularity for weak solutions of equations of the form

$$\text{(1)} \qquad \operatorname{div} A(x,t,u,u_x) + B(x,t,u,u_x) = u_t$$

where A and B are, respectively, vector and scalar valued Baire functions defined on $\Omega \times R^1 \times R^n$, where Ω is an arbitrary open subset of $R^{n+1}(x,t)$. Closely associated with the problem of determining under what condition a boundary point is regular for weak solutions is the question of finding an optimal condition in order that a compact set $K \subset \Omega$ be removable for solutions of (1). Answers to both of these questions are described below in terms of classical parabolic capacity.

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The functions A and B introduced in equation (1) are required to satisfy the following structure conditions:

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(2) \qquad \begin{aligned}
|A(x,t,u,w)| &\le a_0|w| + a_1|u| + a_2 \\
|B(x,t,u,w)| &\le b_0|w|^2 + b_1|w| + b_2|u| + b_3 \\
w \cdot A(x,t,u,w) &\ge c_0|w|^2 - c_1|u|^2 - c_2
\end{aligned}
$$

where a_0, b_0, and c_0 are nonnegative constants with $c_0 > 0$. The remaining coefficients are nonnegative measurable functions in Ω that lie in the following Lebesgue classes:

$$a_1, a_2, b_1 \in L_{2,loc}(\Omega)$$

and

$$b_2, b_3, c_1, c_2 \in L_{1,loc}(\Omega) .$$

Under various assumptions on the structure (2), interior regularity, i.e., Hölder continuity, of weak solutions of (1) has been established by several authors, cf., [KO], [LSU], [AS], [T]. Landis, [LA], announced a Wiener-type criterion for boundary of regularity of solutions to the heat equation, although a complete development of his results has apparently never appeared. Other results concerning boundary regularity of linear parabolic equations include [E], [L1], [L2], [PI], [EK], [PE].

For a general development of removability results in terms of capacities for a wide class of linear (including parabolic) equations the reader is referred to [HP]. Edmunds and Peletier [EP] have also considered the problem of removability for weak solutions of (1); however, the capacity they employ is more restrictive than classical parabolic capacity which we employ below. Indeed, our result is optimal for the class of equations under consideration, because sets of positive classical parabolic capacity are obviously not removable for the heat equation.

2. Main Results

If $U \subset \Omega$ is an open set, a bounded function u whose partial derivatives $\frac{\partial u}{\partial x^j}$, $j = 1, 2, \ldots, n$, belong to $L_{2,loc}(U)$ is said to be a <u>weak solution</u> of (1) in U if

$$\int_U\int \{-\phi_t u + A(x,t,u,u_x)\cdot\phi_x - B(x,t,u,u_x)\phi\}dxdt = 0$$

for all $\phi \in C_0^\infty(U)$. The fundamental solution of the heat operator $H = \frac{\partial}{\partial t} - \Delta$ is defined by

$$G(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp(-|x|^2/4t), & t > 0 \\ 0, & t \leq 0. \end{cases}$$

For any set $E \subset R^{n+1}$, the <u>classical parabolic capacity</u> is defined by

$$C(E) = \sup\{\mu(R^{n+1})\}$$

where the supremum is taken over all nonnegative measures μ supported in E whose potential, $G * \mu$, is everywhere bounded above by 1. If $K \subset R^{n+1}$ is a compact set with $C(K) = 0$, it can be shown, [GZ1], that there is a sequence of smooth functions $\{\eta_j\}$ with the following properties:

$$0 \leq \eta_j \leq 1$$

$$\eta_j = 0 \text{ on a nghd of } K$$

$$\| \nabla \eta_j \|_2 \to 0 \text{ as } j \to \infty ,$$

$$\| H^{\vee} \eta_j \|_1 \to 0 \text{ as } j \to \infty ,$$

$$\eta_j \to 1 \text{ a.e. in } R^{n+1} \text{ as } j \to \infty ,$$

where $H^{\vee} = -\frac{\partial}{\partial t} - \Delta u$ is the adjoint to the heat operator. This information is critical in establishing the following result, [GZ1].

Theorem. Let K *be a relatively closed subset of an open set* $\Omega \subset R^{n+1}$ *. Let* $u \in L_{\infty,loc}(\Omega)$ *with* $\frac{\partial u}{\partial x_j} \in L_{2,loc}(\Omega - K)$ *for* $j = 1, 2, \ldots, n$ *, and suppose that* u *is a weak solution of* (1) *in* $\Omega - K$ *. If* $C(K) = 0$ *, then*

$$\text{(i)} \quad \frac{\partial u}{\partial x_j} \in L_{2,loc}(\Omega) , \quad j = 1, 2, \ldots, n ,$$

and

(ii) u *is a weak solution of* (1) *in* Ω .

For the purpose of describing boundary regularity results, let $\Omega \subset R^{n+1}$ be an arbitrary open set and let $z_0 = (x_0, t_0) \in \partial\Omega$. For $\alpha > 0$, let

$$R_\alpha(r) = B(x_0, r) \times (t_0 - \tfrac{3}{4}\alpha r^2, \; t_0 + \tfrac{1}{4}\alpha r^2)$$

where $B(x_0, r)$ is the ball in R^n with center x_0 and radius r . We associate with $R_\alpha(r)$ a subcylinder $R^*_\alpha(r) = B(x_0, \frac{1}{2}) \times (t_0 - \frac{2}{3}\alpha r^2, \; t_0 - \frac{1}{3}\alpha r^2)$. Let u be a function all of whose partial derivatives are in $L_2(\Omega)$, i.e., u is in the Sobolev space $W_2^1(\Omega)$. If $z_0 \in \partial\Omega$ and $\ell \in R^1$, we say that

$$u(z_0) \le \ell \quad \textit{weakly}$$

if for every $k > \ell$ there is an $r > 0$ such that $\eta(u - k)^+ \in W^1_{2,0}(\Omega)$ whenever $\eta \in C_0^\infty[B(z_0, r)]$. The following theorem comes from [GZ2] and [Z].

Theorem. *Let* $u \in W_2^1(\Omega)$ *be a weak subsolution of* (1) *such that* $u(z_0) \leq \ell$ *weakly.* *If for some* $\alpha > 0$,

$$(3) \qquad \int_0^1 \frac{C[R_\alpha^*(r) - \Omega]}{C[R_\alpha^*(r)]} \frac{dr}{r} = \infty ,$$

then

$$\limsup_{\substack{z \to z_0 \\ z \in \partial\Omega}} u(z) \leq \ell \ .$$

In the case $\Omega = U \times [0,T]$ where U is an arbitrary open set in R^n and $x_0 \in \partial\Omega$, then it is shown in [Z] that (3) holds if and only if x_0 is a regular point for the Laplacian. We conclude with the following.

Corollary. *If* $u \in W_2^1(\Omega)$ *is a weak subsolution of* (1), *then* u *is upper semicontinuous on* Ω .

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