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Maximoff's Theorem

The main purpose of this note is to give a proof of a theorem of Maximoff [M] (according to which for every Darboux function f in the first class of Baire on R there is a homeomorphism h of R onto itself such that foh is a derivative)* We shall prove a bit more general result (Theorem 2, cf. also Remark 1) since it does not require any significant change of the technique. It is possible to generalize the result of Remark 1 to (countable families of) $\overline{R}$-valued functions; this will be done by different methods in a separate paper.

A nonnegative locally finite non-atomic Borel regular measure on R (the set of all real numbers) will be simply called a measure. A measure μ is called positive if $\mu(G) > 0$ whenever G is an open subset of R, $G \neq \emptyset$. If μ is a measure and g is a nonnegative locally μ-integrable function the measure $\nu = g\mu$ is defined by $\nu(A) = \int_A g \, d\mu$; two measures μ, ν are said to be equivalent if there are g,h such that $\mu = g\nu$ and $\nu = h\mu$.

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* Let us remark that the original proof is not only very involved, but also possibly not correct. This has been mentioned several times, e.g. by Lipinski and by Goffman [G].

Let X be a separable metric space, μ a positive measure on R and f be a mapping of R into X. Then

(1) f is said to possess the μ-Denjoy property if $\mu(f^{-1}(G) \cap I) > 0$ provided that $G \subset X$ is open, $I \subset R$ is an interval (open or closed) and $f^{-1}(G) \cap I \neq \emptyset$.

(2) f is said to be μ-approximately continuous if

$$\lim_{y \to x} \frac{\mu^*\{z \in (x,y); \rho(f(z), f(x)) \geq r\}}{\mu(x,y)} = 0 \text{ for every } r > 0$$

and $x \in R$.

(3) f is said to be a μ-Lebesgue function if

$$\lim_{y \to x} \frac{1}{\mu(x,y)} \int^*_{(x,y)} \rho(f(z), f(x))\, d\mu(z) = 0 \text{ for any}$$

$x \in R$.

(4) f is said to be of class M_0 if it is of the first class and $f^{-1}(G) \cap I$ is infinite provided that $G \subset X$ is open, $I \subset R$ is an interval and $f^{-1}(G) \cap I \neq \emptyset$.

(5) f is said to be of class M_1 if it is of the first class and $f^{-1}(G) \cap I$ is uncountable provided that $G \subset X$ is open, $I \subset R$ is an interval and $f^{-1}(G) \cap I \neq \emptyset$.

We shall also denote by χ_A the characteristic function of the set A, by $U(F,\varepsilon)$ the ε-neighborhood of the set F and by λ the Lebesgue measure on R.

Lemma 1. Suppose that

(a) $(a,b) \subset R$ is a bounded open interval, $E \subset R$

(b) ν is a measure on R such that E is ν-measurable and $\nu(E\cap(a,x))>0$ whenever $x\in(a,b)$

(c) α is a monotone nonnegative function on $(a,b]$ such that $\lim_{\substack{t\to a\\ t\in(a,b)}} \alpha(t) = 0$.

Then there exists a ν-integrable nonnegative function φ on R such that

(i) $\{x;\ \varphi(x)\neq 0\}\subset E\cap(a,b)$

(ii) $\int_{(a,b)} \varphi(x)\, d\nu(x) \leq 2\alpha(b)$

(iii) $\int_{(a,t)} \varphi(x)\, d\nu(x) \geq \alpha(t)$ for every $t\in(a,b]$.

Proof. Let $(b_n)_{n=0}^{\infty}$ be a sequence such that $b_0=b$, $b_n\in(a,b_{n-1})$, $\lim_{n\to\infty} b_n = a$ and $\sum_{n=0}^{\infty}\alpha(b_n)\leq 2\alpha(b)$. Put

$$\varphi(x) = \sum_{n=1}^{\infty}\left[\frac{\alpha(b_{n-1})}{\nu(E\cap(a,b_n))}\right]\left[\chi_{E\cap(a,b_n)}\right].$$

Then (i) and (ii) are obvious; let us prove (iii). Let $t\in[b_n,b_{n-1}]$.

Then $\int_{(a,t)} \varphi(x)\, d\nu(x) \geq \alpha(b_{n-1})\,(\nu(E\cap(a,b_n)))^{-1}\nu(E\cap(a,b_n))$

$$\geq \alpha(b_{n-1}) \geq \alpha(t).$$

Lemma 2. Suppose that

(a) $\emptyset \neq F\subset R$ is a compact nowhere dense set, $F\subset E\subset R$

(b) ν is a measure on R such that E is ν-measurable and $\nu(E\cap I)>0$ for every interval I with $I\cap F\neq\emptyset$

(c) ε is a positive number.

Then there exists a ν-measurable nonnegative function ψ

on R such that

(i) $\{x;\ \psi(x) \neq 0\} \subset (E - F) \cap U(F, \varepsilon)$

(ii) $\int \psi(x)\, d\nu(x) < \varepsilon$

(iii) If $x \in F$ then $\lim_{y \to x} \nu((x,y) - F) \cdot \left(\int_{(x,y)} \psi(t)\, d\nu(t) \right)^{-1}$

$= 0$.

Proof. Let $J \supset F$ be a bounded open interval. Put $G = F \cup (R - J)$. Let $(I_n)_{n=1}^{\infty}$ be a sequence of all open intervals contiguous to G. Since $\infty > \nu(G) = \sum_{n=1}^{\infty} \nu(I_n)$, we can find a nondecreasing function ω on $(0, +\infty)$ such that $\lim_{x \to 0+} \omega(x) = 0$, $\lim_{x \to 0+} x^{-1}\omega(x) = +\infty$ and

$\sum_{n=1}^{\infty} \omega(\nu(I_n)) < +\infty$.

For every interval (a,b) contiguous to G such that $a \in F$ we use Lemma 1 with $\alpha(t) = \omega(\nu(a,t))$. Let η be the sum of all functions ψ constructed in this way. Then

(i′) $\{x;\ \eta(x) \neq 0\} \subset E - F$

(ii) $\int \eta(x)\, d\nu(x) \leq 4 \sum_{n=1}^{\infty} \omega(\nu(I_n)) < +\infty$

(iii′) If $x \in F$, $y \in J$, $x \neq y$, then

$$\int_{(x,y)} \eta(t)\, d\nu(t) = \sum_n \int_{I_n \cap (x,y)} \eta(t)\, d\nu(t) \geq$$

(since for $s \in \overline{I}_n = [a_n, b_n]$, $\nu(a_n, s) \leq \nu(x,y)$ and

$$\int_{(a_n,s)} \eta(t)\, dt \geq \omega(\nu(a_n,s)) = \frac{\omega(\nu(a_n,s))}{\nu(a_n,s)} \cdot \nu(a_n,s))$$

$\geq \inf\{t^{-1}\omega(t);\ 0<t \leq \nu(x,y)\} \cdot \sum_{I_n \cap (x,y) \neq \emptyset} \nu(I_n \cap (x,y))$

$= \nu((x,y)-F) \inf \{t^{-1} \omega(t);\ 0<t\leq \nu(x,y)\}.$

Since $\nu((x,y)-F) > 0$ (because F is nowhere dense) it follows $\lim_{y\to x} \nu((x,y)-F) \left(\int_{(x,y)} \eta(t)\, d\nu(t) \right)^{-1} = 0.$

To finish the proof it is sufficient to choose $\delta > 0$ such that $\delta<\varepsilon$ and $\int_{U(f,\delta)} \eta(t)\, d\nu(t) < \varepsilon$ and to put $\psi = \eta\, \chi_{U(F,\delta)}$.

Lemma 3. Let X be a separable metric space, let $f\colon R\to X$ be a mapping of the first class. Then there is a sequence F_n of compact nowhere dense subsets of R such that

(i) If $m<n$ then either $F_m \supset F_n$ or $F_m \cap F_n = \emptyset$

(ii) If $x\in R$ is not a point of continuity of f and $p \in N$ then there exists $n \in N$, $n\geq p$ such that $x \in F_n$ and $\operatorname{diam} f(F_n) < p^{-1}$.

Proof. First note that, for every F_σ-set $M\subset R$ of the first category and every $\varepsilon>0$ there exists a sequence of disjoint compact sets $M_n\subset R$ such that $M = \bigcup_n M_n$ and $\operatorname{diam} f(M_n) < \varepsilon$. To see this, find a sequence of compact sets $H_n\subset M$ such that $M = \bigcup_n H_n$ and $\operatorname{diam} f(H_n)<\varepsilon$ (see [K], chapter 2, §31, II, Theorem 3) and note that, since $X_n = H_n - \bigcup_{k=1}^{n-1} H_k$ are zero dimensional separable metric locally compact spaces, we can write

$X_n = \bigcup_{j=1}^{\infty} M_{n,j}$ where $M_{n,j}$ are disjoint and compact (cf.[K], chapter 2, §26, II, Theorem 1).

Let F be the set of all points of discontinuity of f. Using the preceding observation we can decompose $F = \bigcup_n F_{1,n}$ where $\{F_{1,n}\}_{n=1}^{\infty}$ is a sequence of disjoint compact sets such that diam $f(F_{1,n}) < 1$ for every $n \in N$. By induction we may define, for every $m \in N$, $m > 1$, a sequence $F_{m,n}$ of disjoint compact sets such that $F = \bigcup_n F_{m,n}$, diam $f(F_{m,n}) < m^{-1}$ and every set $F_{m,n}$ is a subset of some $F_{m-1,k}$. The family $\{F_{m,n};\ m,n \in N\}$ can be arranged into a sequence $\{F_m\}$ with the required properties.

<u>Theorem 1.</u> Let X be a separable metric space and let $f: R \to X$ be a mapping of the first class. Let μ be a positive measure on R. Then the following conditions are equivalent.

(1) f has the μ-Denjoy property.

(2) There is a measure ν equivalent to μ such that f is ν-approximately continuous.

(3) There is a measure τ equivalent to μ such that f is

an μ-Lebesgue function.

Proof. (1) $\Rightarrow$(2): Let F_m be compact subsets of R with the properties (i), (ii) of Lemma 3. Using the Luzin theorem we find for every $m \in N$ a compact set $E_m \subset R$ such that

(a) $E_m \supset F_m$

(b) $f|(E_m - F_m)$ is continuous

(c) $E_m \subset f^{-1}(\{x \in X;\ \rho(x, f(F_m)) < m^{-1}\})$

(d) $\mu(E_m \cap I) > 0$ for every interval I intersecting F_m.

Let H_m be the union of all sets F_n such that $n<m$ and $F_n \cap F_m = \emptyset$. If $H_m = \emptyset$, put $\varepsilon_m = 2^{-m}$. If $H_m \neq \emptyset$ first choose $\delta_m > 0$ such that $H_m \cap \overline{U(F_m, \delta_m)} = \emptyset$ and put

$\varepsilon_m = \min(2^{-m}, \delta_m, 2^{-m} (\inf\{\rho(x,y);\ x \in H_m,\ y \in \overline{U(F_m, \delta_m)}\})^2)$.

According to Lemma 2 with $F = F_1$, $E = E_1$, $\varepsilon = \varepsilon_1$, and $\nu = \mu$ we construct a μ-integrable function ψ_1 with the properties (i)-(iii) of Lemma 2 and put $\mu_1 = \mu + \psi_1 \mu$, $\nu_1 = \psi_1 \mu$. By induction we construct sequences of measures $\{\mu_m\}$, $\{\nu_m\}$ and a sequence $\{\psi_m\}$ of μ_{m-1}-integrable functions such that

(i) $\{r \in R;\ \psi_m(r) \neq 0\} \subset (E_m - F_m) \cap U(F_m, \varepsilon_m)$

(ii) $\psi_m \geq 0$ and $\int_R \psi_m(r)\, d\mu_{m-1}(r) < \varepsilon_m$

(iii) $\mu_m = \mu_{m-1} + \psi_m \mu_{m-1}$, $\nu_m = \psi_m \mu_{m-1}$.

(iv) If $r\in F_m$ then $\lim_{s\to r} \mu_{m-1}((r,s)-F_m)\cdot(\gamma_m(r,s))^{-1}=0$.

Then $\mu_m = (\prod_{i=1}^{m} (1+\psi_i))\cdot\mu$ and $\mu_m(I)\le\mu_{m-1}(I)$ $+ 2^{-m}$ for every interval $I\subset R$. Hence $\int_I \prod_{i=1}^{m} (1+\psi_i)d\mu$ $\le \mu(I) + \sum_{i=1}^{m} 2^{-i}$, thus the function $\psi=\prod_{i=1}^{\infty}(1+\psi_i)$ is locally μ-integrable. Put $\nu = \psi\mu$.

We prove that the function f is ν-approximately continuous at every point $r\in R$. Since this is obvious if f is continuous at r, suppose that r is a point of discontinuity of f. Let $p\in N$, $E = f^{-1}(X-U(f(r),p^{-1}))$. Find $m\in N$, $m\ge 2p$ such that $r\in F_m$ and diam $f(F_m) <$ $<(2p)^{-1}$. If $n\ge m$ and $\gamma_n(E\cap(r,s)) > 0$ then $F_n\cap F_m = \emptyset$ and $\gamma_n(E\cap(r,s)) \le \gamma_n(R) \le \varepsilon_n \le$ $2^{-n}(\mu(r,s))^2 \le 2^{-n}\mu(r,s)\,\nu(r,s)$. Hence $\gamma_n(E\cap(r,s))$ $\le 2^{-n}\mu(r,s)\nu(r,s)$ for every $s\in R$, consequently

$$\nu(E\cap(r,s)) = \sum_{n=m}^{\infty} \gamma_n(E\cap(r,s)) + \mu_{m-1}(E\cap(r,s))\le$$

$$\le \sum_{n=m}^{\infty} 2^{-n}\mu(r,s) + \mu_{m-1}(E\cap(r,s)) \le$$

$$\le [\mu(r,s) + \mu_{m-1}(E\cap r,s)\cdot(\gamma_m(r,s))^{-1}]\cdot\nu(r,s).$$

Since $\lim_{s\to r} [\mu(r,s) + \mu_{m-1}(E\cap(r,s))\cdot\gamma_m(r,s)^{-1}] = 0$, the preceding inequality implies the result.

(2) ⇒(3). Choose $x \in X$ and put $g(r) = \rho(x, f(r))$ and $\eta = (1+g)^{-1}\nu$. Let $r \in R$ and let $f_r(s) = \rho\ (f(r), f(s))$. Then the functions $(1+g)^{-1}$ and $f_r(1+g)^{-1}$ are bounded ν-approximately continuous, hence $\lim_{s\to r} (\eta(r,s))^{-1} \cdot \int_{(r,s)} f_r(t)\, d\eta\ (t) =$

$$\lim_{s\to r} \left((\nu(r,s))^{-1} \cdot \int_{(r,s)} (1+g(t))^{-1} d\nu(t) \right)^{-1} .$$

$$((\nu\ (r,s))^{-1} \cdot \int_{(r,s)} f_r(t)(1+g(t))^{-1} d\nu\ (t)) = 0.$$

(3)⇒(2)⇒(1) is obvious.

<u>Theorem 2.</u> Let f be a mapping of R into a separable metric space X. Then the following conditions are equivalent.

(1) f is of class M_0.

(2) f is of class M_1.

(3) f is of the first class and there exists a positive measure μ such that f has the μ-Denjoy property.

(4) There is a positive measure μ such that f is μ-approximately continuous.

(5) There is a positive measure μ such that f is a μ-Lebesgue function.

(6) There is a homemorphism h of R onto itself such that $f \circ h$ is λ-approximately continuous.

(7) There is a homeomorphism h of R onto itself

such that $f\circ h$ is ν-Lebesgue function.

Proof. (1)⇒(2). For every $r \in R$ the real-valued function $f_r(s) = \rho(f(r),f(s))$ is of class M_0 hence it is of class M_1 (see [Z]). Thus, for every $\varepsilon > 0$ and $s \neq r$, the set $f^{-1}(U(f(r),\varepsilon))\cap(r,s) = (r,s)\cap \cap f_r^{-1}(-\varepsilon,\varepsilon)$ is uncountable.

(2)⇒(3). First note that for every uncountable Borel set $B\subset R$ there is a finite measure on R such that the measure of B is positive. To prove this, choose two nowhere dense nonempty compact sets $P,Q\subset R$ without isolated points such that $\lambda(P)>0$ and $Q\subset B$ (the existence of Q follows from [K], chapter 3, §37, I, Theorem 3). Let h be a homeomorphism of Q onto P (see [K], chapter 4, §45, II, Theorem 1). Put $\nu(E) = \lambda(h(E\cap Q))$ for every Borel set $E\subset R$.

Let $\{G_n;\ n\in N\}$ be a countable basis of open sets of X and let $\{I_n;\ n\in N\}$ be a sequence of all rational intervals. For every $m,n\in N$ for which the set $E_{m,n} = f^{-1}(G_m)\cap I_n$ is nonempty (hence uncountable) find a measure $\mu_{m,n}$ such that $\mu_{m,n}(E_{m,n}) > 0$ and $\mu_{m,n}(R) = 2^{-m-n}$. It is sufficient to consider

$$\mu = \sum_{m,n} \mu_{m,n}.$$

(3)⇒(4)⇒(5) follows directly from Theorem 1.

(5)⇒(7). Suppose that g is a positive real-valued continuous function on R and put $\nu = g\mu$.

Then $\lim_{s\to r} (\nu(r,s))^{-1} \cdot \int_{(r,s)} \rho(f(r), f(t))\, d\nu(t) \le$

$\le \lim_{s\to r} (\inf \{g(t); t\in(r,s)\}\mu(r,s))^{-1} \cdot (\sup \{g(t);$

$t\in(r,s)\} \cdot \int_{(r,s)} \rho(f(r),f(t))\, d\mu(t)) = 0.$

Hence, considering $g\mu$ with a suitable g instead of μ if necessary, we may assume that $\mu(0,+\infty) = +\infty$ and $\mu(-\infty,0) = +\infty$.

Put $H(x) = \mu(0,x)$ for $x\ge 0$ and

$H(x) = -\mu(x,0)$ for $x<0$.

Then H is a homeomorphism of R onto R, let h be its inverse. Then $\int \psi(t)\, d\mu(t) = \int \psi(h(t))\, d\lambda(t)$ for every nonnegative Borel function on R.

Let $r \in R$, $u = h(r)$. Then

$\lim_{s\to r} (\lambda(r,s))^{-1} \cdot \int_{(r,s)} \rho(f(h(r)), f(h(t)))\, d\lambda(t) =$

$\lim_{s\to r} (\mu(u,h(s)))^{-1} \cdot \int_{(u,h(s))} \rho(f(u),f(t))\, d\mu(t) = 0.$

(7)⇒(6) is obvious.

(6)⇒(1). For every $x\in X$ the function $g(r) = \rho(x,f(h(r)))$ is approximately continuous, hence it is of class M_0 (see [Z]). The proof now follows from

the equalities $f^{-1}(U(x,s)) = h[(f\circ h)^{-1}(U(x,s))] =$
$= h[g^{-1}(-s,s)]$.

<u>Corollary</u>. (Maximoff's Theorem) Let f be a real-valued function on R. The following conditions are equivalent.

(1) f is a Darboux function of the first class.

(2) There is a homeomorphism h of R onto R such that $f\circ h$ is a derivative.

Proof. (1)⇒(2) follows directly from the implication (1)⇒(7) in Theorem 2 (with X=R).
(2)⇒(1) follows from the well-known fact that any derivative is a Darboux function of the first class.

<u>Remark 1</u>. If $\{f_1,\dots,f_n\}$ is a finite family of real-valued functions on R then Theorem 2 (with $X=R^n$) gives necessary and sufficient conditions for the existence of a homeomorphism h of R onto itself such that all functions $f_i\circ h$ are (λ-)Lebesgue functions. On the other hand, this condition is not necessary for the existence of a homeomorphism h such that all functions $f_i\circ h$ are derivatives. An obvious necessary condition is that every linear combination of f_i is a Darboux function of the first class. Is this condition also sufficient?

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