Real Analysis Exchange Vol. 5 (1979-80)

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A Generalization of Absolute Continuity

The paper which follows is concerned with continuous functions which satisfy the following property:

> A function f is said to satisfy condition (M) providing f is absolutely continuous on any set E on which it is of bounded variation.

The condition (M) appears to be relevant to the theory of the integral (cf. [2]). A function f is said to satisfy Lusin's condition (N) provided the image under f of each set of measure 0 is of measure 0. Since bounded variation on a set along with Lusin's condition (N) implies absolute continuity, condition (N) implies condition (M). The example given below shows that functions which satisfy (M) need not satisfy (N) and in fact need not satisfy Banach's condition T_2 (for almost every y, $f^{-1}(y)$ is at most countable). Of course, condition (N) implies T_2 and it is this fact that leads to the proof that functions satisfying (N) must be differentiable on a set of positive measure in every interval [3, p. 286]. Actually, functions satisfying (M) also have this property, as is shown by Theorem 3.

Before proceeding, it is worthwhile to prove the following theorem:

<u>Theorem 1</u>. <u>A continuous function</u> f <u>satisfies</u> (M) <u>if and only if</u> f <u>is absolutely continuous on any set</u> E <u>on which</u> f <u>is monotone</u>.

<u>Proof</u>. Clearly, (M) implies the second property. Suppose a function f satisfies the second property and E is a set on which f is of bounded variation. Then $f|_E$ can be extended to a function F defined on the entire line so that F is a linear on intervals contiguous to $\bar{E}$. By De la Valee Poussin's Theorem [3, p. 125-127] if N is the set of points where $F'(x)$ does not exist (finite or infinite) then the length of the graph of F on N is 0 and thus $|N| = |F(N)| = 0$. But if D is the set of points where $F'(x)$ exists, it is known that F satisfies condition (N) on D [3,p.227]. If $E_{+\infty} = \{x|F'(x) = +\infty\}$ and $E_{-\infty} = \{x|F'(x) = -\infty\}$, then every x belongs to either D, N, $E_{-\infty}$, or $E_{+\infty}$. If $|F(E_{+\infty})|>0$ then a subset A of $E_{+\infty}$ can be found on which F is monotone increasing and $|F(A)|>0$. To see this, let

$$E_n = \{x \mid \text{if } 0<|h|\le 1/n, \frac{F(x+h) - F(x)}{h} > 1\}$$

and let

$$E_{in} = [\frac{i}{n}, \frac{i+1}{n}] \cap E_n .$$

Then F is monotone increasing on each set E_{in} and $E_{+\infty} \subset \bigcup_{i,n} E_{in}$. However, if f is absolutely continuous on each set for which f is monotone, $|F(E_{in} \cap E_{+\infty})| = 0$ and thus $|F(E_{+\infty})| = 0$. Similarly $|F(E_{-\infty})| = 0$ and thus F satisfies (N). It follows that f satisfies (M) .

Example: There exists a continuous function F such that F satisfies (M) but F does not satisfy (N) and in fact F does not satisfy Banach's condition T_2 .

Construction. If $x \in [0,1]$, x can be expressed uniquely as $\Sigma x_i/32^i$ where $0 \le x_i \le 31$, each x_i is a whole number and there exists N so that $x_i \ne 31$ for all $i>N$. Let $P = \{x \mid x = \Sigma x_i/32^i$ with all x_i even$\}$. Then P is a perfect set. Define $F(x)$ on P as follows:

$F(x) = \sum b_i/8^i$, where $b_i = 1$ if $x_i = 0$ or 16,
$b_i = 0$ if $x_i = 2$ or 18, $b_i = 3$ if $x_i = 4$ or 20,
$b_i = 2$ if $x_i = 6$ or 22, $b_i = 5$ if $x_i = 8$ or 24,

$b_i = 4$ if $x_i = 10$ or 26, $b_i = 7$ if
$x_i = 12$ or 8,
and $b_i = 6$ if $x_i = 14$ or 30.

Then F is continuous on P. Define F on $[0,1]$ by extending it linearly to the intervals contiguous to P in such a fashion as to make F continuous. Then F satisfies (N) on $[0,1]\setminus P$. In order to show that F satisfies (M), by Theorem 1, it suffices to show that if $A\subset P$ and F is monotone on A, F satisfies (N) on A. Suppose F is monotone non-decreasing on $A\subset P$. By observing the behavior of $F(x)$ it is apparent that no 8 points x^i, $i = 1,2,\ldots,8$ of A can have distinct x_1's. For example if $x_1^1<x_1^2<\ldots<x_1^8$ and $b_1^1 = 0$, $x_1^1 = 2$ or 18. If $b_1^2 = 1$ since $x_1^2\neq 0$, $x_1^2 = 16$ and x_i^1 must equal 2. If $b_1^3 = 2$, $x_1^3 = 22$ and then b_1^4 cannot equal 3. Thus $|F(A)|\leq 7/8$. Similarly no 8 points with the same x_1 can have distinct x_2's and thus $|F(A)|\leq(7/8)^2$. Continuing this argument leads to the fact that $|F(A)| = 0$ and that F is absolutely continuous on A. A similar analysis shows that F is absolutely continuous on any set B on which F is monotone non-increasing. To see that F does not satisfy Banach's condition T_2, it is sufficient to note that for each $y\in[0,1]$ if $y = \Sigma b_i/8^i$ then the set of

$x = \Sigma x_i/32^i \epsilon P$ (where the x_i follow the rules defining F on P) which map onto y is a perfect subset of [0,1] .

In [1] it is shown that there are two functions f_1, f_2 which satisfy (N) such that $f_1 - f_2$ is a monotone singular function. The following theorem shows that for each function f satisfying (M) but not (N) there is a $g \epsilon (M)$ such that $f - g$ is monotone singular.

<u>Theorem 2</u>. <u>If</u> f <u>is continuous and satisfies</u> (M) <u>but does not satisfy</u> (N), <u>there exists a monotone non-decreasing singular function</u> h <u>such that</u> f + h <u>satisfies</u> (M) .

Proof. Suppose f is defined on [0,1] and satisfies (M) but not (N) . Then there is a perfect set P with $|P| = 0$ such that $|f(P)|>0$. Let $h(x) = |f(P\cap(0,x))|$. Then h is non-decreasing and is constant on each interval contiguous to P . Let $A = \{x \epsilon P | \exists t<x$ with $t \epsilon P$ and $f(t) = f(x)\}$. Then $|f(P\cap(0,x))| = |f(A\cap(0,x))|$ and f is one to one on A . Thus for any interval (a,b),

$$h(b) - h(a) = |f(A\cap(0,b))| - |f(A\cap(0,a))| = |f(A\cap(a,b))| .$$

Consequently, for G open, $|h(G)| = |f(G \cap A)| \leq |f(G)|$ and therefore for closed sets F, $|h(F)| \leq |f(F)|$. Now suppose there is a $P' \subset P$ such that $f + h$ is bounded variation but not absolutely continuous on P'. Then f is bounded variation on P' which implies f is absolutely continuous on P' and $|f(P')| = 0$. But for closed sets $F \subset P'$, $|h(F)| \leq |f(F)|$ and thus h is absolutely continuous on P'. Consequently, $f + h$ is also absolutely continuous on P'. Thus $f + h$ satisfies (M) on P. Since h is constant on intervals contiguous to P, it follows readily that $f + h$ satisfies (M).

<u>Theorem 3</u>. <u>If</u> f <u>is a continuous function on an interval</u> I <u>and if</u> $\{x \in I \mid f'(x)$ <u>exists</u>$\}$ <u>has measure</u> 0, <u>then there is a perfect set</u> P <u>with</u> $|P| = 0$ <u>such that</u> f <u>is increasing on</u> P <u>and</u> $|f(P)| > 0$.

<u>Note</u>. Theorems 1 and 3 imply that each continuous f satisfying (M) is differentiable on a set of positive measure in every interval.

<u>Proof</u>. Suppose there is a function f defined on an interval I with $\{x \mid f'(x)$ exists$\}$ of measure 0 and there does not exist a $P \subset I$ with $|P| = 0$,

$|f(P)|>0$ and f increasing on P. Choose ε_i with $1>\varepsilon_i>0$ such that $\Pi(1-\varepsilon_i)^2>\frac{1}{2}$.

Let a be a relative minimum of f on I and let $b\in I$ with $b>a$ and $f(b)>f(x)>f(a)$ for $x\in(a,b)$. Let $F_o = [a,b]$. Let $E_o' = \{x\in F_o \mid \exists t\in F_o$ with $t<x$ and $f(t) = f(x)\}$, then f is non-decreasing on $\bar{E}_o'$ and $f(\bar{E}_o') \supset [f(a), f(b)]$. Let $G_o = \cup(x-r_x, x+r_x)$ where the union is over all $(x-r_x, x+r_x)$ with $x\in E_o'$ and $|f((x-r_x, x+r_x))\cap E_o'| = 0$. Let $E_o = E_o' \setminus G_o$ then, for each open interval J, $E_o\cap J$ is either empty or of positive measure. (Otherwise, $E_o\cap J$ contains a set which can serve as P.) Let P_1 be a perfect subset of E_o so that at each point of P_1, $\underline{f}^+ = -\infty$, and so that $|f(P_1)|>|f(E_o)|(1-\varepsilon_1)$. This can be done since at almost every point of E_o, $\underline{f}^+ = -\infty$ and the image of the set $E_o\cap(\underline{f}^+ = -\infty)$ is almost all of the image E_o [3, p. 271]. Let $\{I_n\}$ be the set of intervals contiguous to P_1 in F_o and choose N so large that $\sum_{n>N} |I_n|<\varepsilon_1$. For each $x\in P_1$ which is not a left hand end point of any I_n with $n\leq N$, choose $c_x\in\bigcup_{n>N} I_n$ so that $f(c_x)<f(x)$. Let d_x be the right hand end point of interval I_n which contains c_x. Then the intervals $[f(c_x), f(d_x)]$ cover all

but a finite collection of points $f(P_1)$. Select from a set of intervals $\{[c_x, d_x]\}$ a finite subcollection $\{[c_i, d_i]\}$ so that $|\cup[f(c_i), f(d_i)]|>|f(P_1)|(1-\varepsilon_1)$ and so that no interval $[f(c_i), f(d_i)]$ is contained in another interval $[f(c_j), f(d_j)]$.

Arrange the c_i, d_i so that $c_1<d_1<c_2<d_2<\ldots<c_k<d_k$. Since $f(d_1)<f(d_2)<\ldots<f(d_k)$, points $u_1, u_2 \ldots u_k, v_1, v_2 \ldots v_k$ can be chosen inductively as follows with $[u_i v_i]\subset[c_i d_i]$:

let $y_i = \inf\{f(x)|x\in[c_i d_i]\}$,

let $u_1 = \sup\{x\in[c_1 d_1]|f(x) = y_1\}$,

let $v_1 = \inf\{x\in[c_1 d_1]|x>u_1$ and $f(x)=f(d_1)\}$,

let $u_{i+1} = \sup\{x\in[c_{i+1}, d_{i+1}]|f(x) = \max(y_{i+1}, f(d_i))\}$,

let $v_{i+1} = \inf\{x\in[c_{i+1}, d_{i+1}]|x>u_{i+1}$ and $f(x) = f(d_{i+1})\}$.

Note that if $x_1\in[u_i v_i]$ and $x_2\in[u_{i+1}, v_{i+1}]$, it follows that $f(x_1)\le f(x_2)$.

Let $F_1 = \cup[u_i, v_i]$.

Let $E_1' = \{x\in F_1|\exists t\in F_1$ with $t<x$ and $f(t) = f(x)\}$

then f is non-decreasing on E_1' .

Let $G_1 = \cup(x-r_x, x+r_x)$ where the union is over all $(x-r_x, x+r_x)$ with $x \in E_1'$ and

$|f((x-r_x, x+r_x) \cap E_1')| = 0$.

Let $E_1 = E_1' \setminus G_1$.

Then $|f(E_1)| = |f(E_1')|$ and, for each open interval J, $\bar{E}_1 \cap J$ is either empty or of positive measure. (Otherwise, $\bar{E}_1 \cap J$ would contain a P .) Note that $F_o \supset E_o \supset P_1$, $P_1 \cap \overset{o}{F}_1 = \phi$,

$F_o \supset F_1 \supset E_1$, $|F_1| < \varepsilon_1$,

and $|f(E_1)| > |f(E_o)|(1-\varepsilon_1)^2$.

In general suppose P_n , F_n , and E_n have been chosen so that $P_n \subset E_{n-1} \subset F_{n-1}$, $P_n \subset (\underline{f}^+ = -\infty)$, F_n is the union of a finite collection of closed intervals each of whose interior is contained in an interval contiguous to P_n , $|F_n| < \varepsilon_n$, f is non-decreasing on $\bar{E}_n$, $|f(E_n)| > |f(E_{n-1})|(1-\varepsilon_n)^2$, and for each open interval J , $\bar{E}_n \cap J$ is either empty or has positive measure. Suppose further that if x_1 is an interval of F_n and x_2 is in a different interval and $x_1 < x_2$, then $f(x_1) \le f(x_2)$. Then, by following the steps of the construction of P_1, F_1, and E_1, the sets P_{n+1}, F_{n+1}, and E_{n+1} can be obtained so that the above properties hold for them with n replaced by $n+1$.

Finally, let $P = \cap F_n$. Since $|F_n| < \varepsilon_n$, $|P| = 0$.

Let $E = \cap f(E_n)$.

Then $|E| > |f(E_o)|(1-\varepsilon_1)^2 \ldots (1-\varepsilon_n)^2 \ldots$

and $|E| > |f(E_o)| - \frac{1}{2} > 0$.

If $y \in E$, $\exists x_n \in F_n \ni f(x_n) = y$ and hence there is a limit point of the x_n , say x_o , with $x_o \in \cap F_n$. By the continuity of f , $f(x_o) = y$. Hence, $f(P) \supset E$ and $|f(P)| > 0$. It remains to show that f is non-decreasing on P . But if $x_1, x_2 \in P$ with $x_1 < x_2$ and if $x_2 - x_1 > \varepsilon_n$ then x_1 and x_2 are in distinct intervals of F_n and hence

$f(x_1) \le f(x_2)$. The existence of such a P contradicts our hypothesis and thus the theorem is proved.

REFERENCES

1. Foran, J., Differentiation and Lusin's Condition (N), Real Analysis Exchange, Vol. 3, #1, p. 34-37.

2. __________, On Extending the Lebesgue Integral (submitted)

3. Saks, S., Theory of the Integral, New York, 1937

Received November 1, 1979