TOPICAL SURVEY *Real Analysis Exchange Vol. 5 (1979-80)*

CURRENT TRENDS IN DIFFERENTIATION THEORY

by Andrew Bruckner*

Since the appearance of Zahorski's article [76] in 1950, a great deal of work has been accomplished concerning the differentiation of real functions. Much of this work has, in fact, been accomplished during the last decade, and even this most recent work has involved a number of different directions of inquiry.

Our purpose here is to discuss some of these directions of inquiry in a largely expository manner. While we state some of the recent results in precise form, our purpose is more to impart the flavor of the subject than it is to provide a complete up-to-date catalogue of the results. For that reason, our style is often informal, none of the chapters is intended to fully summarize the present state of knowledge, and we have omitted certain topics of current interest. Notable among the omitted topics are the important works by Laczkovich and Petruska on extensions to, approximations of, and separation by, derivatives; by Garg on the delicate differentiability structure of functions; by S. Marcus and others on the stationary and determining sets of various classes of functions; and by various authors on certain generalized derivatives.

*The author was supported in part by an NSF grant.

I. DIFFERENTIATION AND CHANGES OF SCALE

Perhaps the simplest example of a continuous function which fails to be differentiable at some point is the function $F(x) = |x|$. The corner appearing on the graph of F disappears however, when F is composed with the homeomorphism $h(x) = x^3$ of $[-1,1]$ onto itself: that is, the function $(F \circ h)(x) = |x^3|$ is everywhere differentiable whereas F is not. In short, we have been able to create differentiability via a homeomorphic change of variables.

In recent years there have been a number of studies of the effect of changes of scales on various classes of functions, particularly classes related to differentiation theory. Most of the results of these studies have intrinsic value to the study of differentiation, of course, but they also can be applied in a variety of ways. For motivational reasons, we shall begin our discussion with statements of a few of these results and some applications. We do this in Section 1, below. Then in Section 2 we consider some general questions; we devote Section 3 to a summary of known results and end with a brief discussion of related questions.

1. Applications of Theorems on Changes of Scale.

To give an indication of the kinds of applications to which these change-of-scale theorems give rise, let us begin with a single such theorem (and some variants) and discuss several applications.

Our starting point is the following theorem [20], [17].

<u>Theorem</u>. A function F defined on the interval [0,1] can be transformed into one with a bounded derivative via a homeomorphic change of variables if and only if F is continuous and of bounded variation on [0,1].

We discuss several applications of this theorem and its modifications.

a) Differentiable Cantor-like Functions.

The Cantor function fails to be differentiable at any point of the Cantor set. Yet, this function is continuous and of bounded variation and can therefore be transformed into one with a bounded derivative. The transformed function is, of course, a Cantor-like function: that is, it is constant on each interval contiguous to some nowhere-dense perfect set, but not constant on any open interval containing points of that set. And it is differentiable, with a bounded derivative:

b) Nowhere Monotone Functions.

A continuous nowhere-differentiable function cannot be monotonic on any interval. Almost one hundred years ago, the problem arose of determining whether or not a <u>differentiable</u> function could be nowhere monotonic. Around the turn of the century, a number of authors published examples of such functions. Unfortunately, their complicated constructions contained errors. Finally, in 1915, Denjoy [21] presented a (correct) lengthy study of such functions. Needless to say, his constructions were quite complicated.

The machinery available to contemporary mathematicians allowed for simpler proofs of existence of differentiable nowhere monotonic functions. Thus, Zahorski [76] obtained such functions by using his deep results about derivatives, Goffman [31] based a proof of existence on properties of the density topology, Petruska and Laczkovich [60] used work they had developed concerning extensions of derivatives and Weil [74] based a proof on the Baire Category Theorem. Interesting "elementary" proofs were recently advanced by Katznelson and Stromberg [40] and by Blazek, Borák and Maly [6].

Suppose we wished to construct such a function from scratch. How could we proceed? A natural attempt would be to try to make judicious use of Pompeiu derivatives. Pompeiu [61] constructed a differentiable increasing function F such that $F' > 0$ on a dense set and $F' = 0$ on another dense set. Perhaps one can construct two such functions F and G so that F-G has the desired properties. One can, but it isn't easy! There just are too many things that have to be controlled simultaneously. Our theorem is helpful here, however [59]. Let A be a measurable set with the property that both A and its complement $\tilde{A}$ intersect each subinterval of [0,1] in a set of positive measure. Let $F(x) = \int_0^x \chi_A(t)dt$, $G(x) = \int_0^x \chi_{\tilde{A}}(t)dt$. Then $F' = 1$ on a dense set, $G' = 1$ on a different dense set and the function F-G is an absolutely continuous nowhere monotonic function. It isn't differentiable everywhere, of course, but our theorem guarantees the existence of a homeomorphism h of [0,1] onto itself such that $(F-G) \circ h$ is differentiable (with a bounded derivative). It is clear that this function is nowhere monotonic because monotonicity cannot be created or destroyed by a homeomorphic change of scale.

Incidentally, Pompeiu derivatives exhibit a number of interesting properties, and such derivatives are <u>not</u> difficult to construct by elementary means [49], [7]. We mention in passing that Solomon Marcus has pointed out that a bounded Pompeiu derivative furnishes an example of a bounded derivative which fails to be Riemann integrable on any interval and this fact follows from completely elementary considerations (the lower Darboux sums must be 0).

c) Parametric Representation of Rectifiable Curves.

Suppose we are told that a rectifiable curve γ in R_2 has parametric representation $x = x(t)$, $y = y(t)$, $(0 \leq t \leq 1)$ with x and y differentiable with bounded derivatives. What does that say about γ? Does it imply, for example, that γ has a tangent at each point, the tangent being vertical at points at which x' vanishes? The answer is that it says <u>nothing</u>, beyond the obvious (that γ is a continuous rectifiable curve). This follows from a small variant of our theorem [20]. Since γ is a rectifiable curve, it admits a parametric representation with coordinate functions continuous and of bounded variation. The variant of our theorem allows us to choose a homeomorphic change of scale which simultaneously transforms the coordinate functions into ones with bounded derivatives.

d) Level Sets of Functions.

Independently, Fleissner and Foran [25] and Kaplan and Slobodnik [39] proved that a function F can be transformed into a differentiable function via a homeomorphic change of variables if and only if F is continuous and of generalized bounded variation (VBG_*). Because a differentiable function must have almost every level set finite, the same must be true of each continuous function which is VBG_*. This fact is well known [66], but it provides an example of the way a change of scale theorem can be used to establish a structure property for a class of functions when that property is known for an appropriate subclass. The property must, of course, be one that is invariant under homeomorphic changes of variables.

It may be interesting to note for comparison that if we weaken differentiability to differentiability a.e., the result of the paragraph above fails completely. Such a function can have <u>every</u> level set perfect. To see this we invoke another change of variables theorem together with an example of Gillis'.

Gillis [30] constructed a continuous function G for which every level set is perfect. Then, in [13] it was shown that a function G can be transformed into one which is differentiable a.e. if and only if G is continuous on a dense set. The resulting function $G \circ h$ is still a Gillis-like function - (every level set is perfect), and it is differentiable a.e. Note $(G \circ h)'$ vanishes wherever it exists. In this connection, it is interesting to observe that a singular function (i.e., a continuous nonconstant function of bounded variation whose derivative vanishes a.e.), must also have an infinite derivative on some c-dense set.

2. The General Situation.

Let us formulate some questions pertaining to the effects of changes of scale on a class of functions. Let $\mathcal{F}$ be a class of functions defined on, say, the interval $[0,1]$. Let $\mathcal{H}$ denote the class of (increasing) homeomorphisms of $[0,1]$ onto itself and let $\mathcal{F} \circ \mathcal{H}$ denote the set of all functions of the form $f \circ h$ where $f \in \mathcal{F}$ and $h \in \mathcal{H}$.

A systematic study of the effects of $\mathcal{H}$ on $\mathcal{F}$ might include consideration of the following questions:

(i) Does $\mathcal{F} \circ \mathcal{H} = \mathcal{F}$? In other words, is the class $\mathcal{F}$ invariant under homeomorphic changes of variables? The class of continuous functions, the class of functions of bounded variations and the class of functions for which each level set is finite furnish examples for which the answer to our question is affirmative. Usually classes of functions differentiable in some generalized sense and the classes of their (generalized) derivatives are not invariant under such a change of scale.

If the answer to (i) is negative, there are several other questions which arise naturally:

(ii) How can we characterize the class $\mathcal{F} \circ \mathcal{H}$?
Observe that the class $\mathcal{F} \circ \mathcal{H}$ consists of those functions which can be transformed into the class $\mathcal{F}$ by a suitable homeomorphic change of variable. The theorem of the previous section shows that for $\mathcal{F}$ the class of functions with bounded derivatives, $\mathcal{F} \circ \mathcal{H}$ is the class of continuous functions of bounded variation.

(iii) How can we characterize the class
$\mathcal{F}^* = \{f \in \mathcal{F}:\ f \circ h \in \mathcal{F}$ for <u>every</u> $h \in \mathcal{H}\}$?

Here we are calling for a characterization of those functions in $\mathcal{F}$ which remain in $\mathcal{F}$ under all homeomorphic changes of scale. Note that $\mathcal{F}^* \circ \mathcal{H} = \mathcal{F}^*$ in general.

(iv) How can we characterize the class
$\mathcal{H}^* = \{h \in \mathcal{H}:\ f \circ h \in \mathcal{F}$ for all $f \in \mathcal{F}\}$? (i.e., under which homeomorphic changes of variable does $\mathcal{F}$ remain invariant?)

One can, of course, ask the analogous questions for changes of scale of the <u>range</u> of the functions in $\mathcal{F}$. Thus, if we now denote by $\mathcal{H}$ the class of increasing homeomorphisms of R_1 onto itself we can call for characterizations of the classes $\mathcal{H} \circ \mathcal{F}$, $\mathcal{F}^*$ and $\mathcal{H}^*$, where the notations have the obvious meaning.

3. Summary of Recent Work.

Much work pertaining to the questions we discussed above has been accomplished in recent years. We summarize some of this work in the chart below. The numbers appearing in brackets are references to proofs of the assertions on the chart. The numbers appearing in parentheses refer to remarks following the chart. A blank space indicates the entry is either a trivial one or one unknown to us. For additional discussions and some proofs see [28], [7].

We shall use the notation below to represent certain classes of functions related to differentiation theory:

C^1	is the class of continuously differentiable functions.
Δ_b	is the class of differentiable functions with bounded derivatives.
Δ	is the class of differentiable functions.
Δ_{ae}	is the class of functions differentiable a.e.
Δ_{ap}	is the class of approximately differentiable functions.
Δ_{apae}	is the class of functions approximately differentiable a.e.
L	is the class of functions for which each point is a Lebesgue point.
Δ'	is the class of derivatives.
$b\Delta'$	is the class of bounded derivatives.
C_{ap}	is the class of approximately continuous functions.

In addition, the notations C, BV and VBG_* will have their usual meaning.

INNER CHANGES OF SCALE

$\mathcal{F}$	$\mathcal{F} \circ \mathcal{H}$	$\mathcal{F}^*$	$\mathcal{H}^*$
C'	(1)[17]		
Δ_b	C and BV [20][17]		
Δ	C and VBG_* [25][39]		
Δ_{ae}	Continuous on dense set [13](4)		h^{-1} absolutely continuous [29]
Δ_{ap}			
Δ_{apae}	Each interval contains a nonempty perfect set P such that F/P is continuous, [13]		h^{-1} absolutely continuous [29]
L	Darboux Baire 1 [51][53]	C [45][11](9)	
Δ'	Darboux Baire 1 [51][51]	C	(2)[42]
$b\Delta'$	Bounded Darboux Baire 1 [51][53]	C	
C_{ap}	Darboux Baire 1 [52]	C [10]	h^{-1} preserves density points [10](3)

OUTER CHANGES OF SCALE

$\mathcal{F}$	$\mathcal{H} \circ \mathcal{F}$	$\mathcal{F}^*$	$\mathcal{H}^*$
C'			
Δ_b	S' (5) [26],[7]		
Δ	S' (5) [26]		
Δ_{ae}	(8) [4]		
Δ_{ap}			
Δ_{apae}			
L	(6) [35]	Bounded functions in C_{ap}	
Δ'	(7)		Linear [20]
$b\Delta'$	(7)	C_{ap}[75]	Linear [20][12]
C_{ap}	C_{ap}	C_{ap}	$\mathcal{H}$

Remarks About the Chart.

(1) The class $C' \circ \mathcal{H}$ consists of those functions which are continuous, of bounded variation and whose set V of points of varying monotonicity map into a set of measure zero. A point x is said to be a point of varying monotonicity for f if there is no neighborhood of x on which f is constant and no neighborhood on which f is strictly monotonic. For example, for the Cantor function f, V is the Cantor set. Since $f(V) = [0,1]$ in this case, $f \notin C' \circ \mathcal{H}$.

(2) The condition obtained by Laczkovich and Petruska [42] is too complicated to state here. It is closely related to the condition that $1/h'$ be continuous and of bounded variation.

(3) The condition is that if x_0 is a point of density of E, then $h^{-1}(x_0)$ is a point of density of the set $h^{-1}(E)$.

(4) The characterization for $\Delta_{ae} \circ \mathcal{H}$ can be formulated in a manner which allows comparison with the class $\Delta_{apae} \circ \mathcal{H}$ as follows: $F \in \Delta_{ae} \circ \mathcal{H}$ if and only if each interval contains a perfect set P such that F is continuous at each point of P. (For $F \in \Delta_{apae} \circ \mathcal{H}$ we require only relative continuity.) Note that $\Delta_{ae} \circ \mathcal{H}$ contains, in particular, all Baire 1 functions, and $\Delta_{apae} \circ \mathcal{H}$ contains all measurable functions and all functions with the property of Baire.

(5) The condition S' is related to Luzin's condition (N). A function F satisfies condition S' if there is no

interval which is contained in the image under F of sets of arbitrarily small measure.

(6) Hancock's conditions are quite complicated. In addition to the obvious requirement of approximate continuity, each member f of $\mathcal{H} \circ L$ must satisfy certain growth and density-like conditions. In particular, if f is approximately continuous and its set of non-Lebesgue points is denumerable, then $f \in \mathcal{H} \circ L$.

(7) Characterizations of the classes $\mathcal{H} \circ \Delta'$ and $\mathcal{H} \circ b\Delta'$ are not known. The inclusions $\mathcal{H} \circ \Delta' \subset \mathcal{M}_3^*$, and $\mathcal{H} \circ b\Delta' \subset \mathcal{M}_4$ follow from the work of Preiss [63] and Zahorski [76], respectively. Both inclusions are proper.

(8) Bary [4] showed that $C \subset \mathcal{H} \circ \Delta_{ae}$. In the same paper she showed $C \subset \Delta_{ae} \circ \mathcal{H}$.

(9) An interesting application of Maximoff's theorem is due to Lipinski [45]. He expressed $\mathcal{DB}_1$ as a union of sets whose intersection is C.

4. Related Questions.

There are many other questions one could ask concerning the composition $f \circ h$, $(f \in \mathcal{F}, h \in \mathcal{H})$. For example, one could replace $\mathcal{H}$ by a more restrictive class of homeomorphisms or by a larger class of functions. Thus, Tolstoff [70] studied the composition $\Delta \circ D$, where D denotes the diffeomorphisms of [0,1] onto itself, and Laczkovich and Petruska [42] studied the composition $f \circ c$, $f \in \Delta'$, c a convex homeomorphism. Similarly, in [10] one finds a characterization of those homeomorphisms h whose inverses are infinitely differentiable such that $f \circ h \in C_{ap}$ whenever $f \in C_{ap}$.

One of the early researches of these types of questions is the work [75] of Wilkosz. He studied functions f such that both f and f^2 are in $b\Delta'$, and showed that this class consists of the bounded approximately continuous functions. He viewed f^2 as the product $f \cdot f$ because he was interested in products of derivatives, but one can also view f^2 as the composition $s \circ f$ where $s(x) = x^2$. We can view Wilkosz' result as a special case of the theorem that $c \circ f \in b\Delta'$ for c strictly convex and $f \in b\Delta'$ if and only if $f \in C_{ap}$ [12]. It follows from this theorem, for example, that if $0 < k \leq f \leq K < \infty$ and $f \in \Delta'$, then $f^2 \in \Delta'$ if and only if $1/f \in \Delta'$. Also, the equality provides a test for approximate continuity. For example, the fact that the function $f(x) = \sin \frac{1}{x}$ $(f(0) = 0)$ is in $b\Delta'$, but f^2 is not, proves $f \notin C_{ap}$.

One finds in [12] additional theorems which show how totally devastated some derivatives are under <u>every</u> nowhere linear homeomorphism change of scale of the range.

Similar questions involving the structure of $\mathcal{F} \circ \mathcal{H}$ or $H \circ \mathcal{F}$ for other classes of functions have also been answered. See for example [2][19][33] and [15] which deal with Fourier series and Baire 1 equivalent functions, respectively. See also the survey article [28] by Foran for further discussions of change of scale theorems within the class of continuous functions and for statements of some open problems.

As a final remark, we mention that transformations involving inner and outer homeomorphisms simultaneously can also be studied. We can ask the same kind of questions we asked before in connection with classes of the form $\mathcal{H} \circ \mathcal{F} \circ \mathcal{H}$ where the notation has obvious meaning. For example, if $\mathcal{F} = \Delta$ or Δ_b, the classes $\mathcal{H} \circ \mathcal{F} \circ \mathcal{H}$ (in both cases) consist of those continuous functions f such that (*) $\{y: f^{-1}(y)$ is finite$\}$ is c-dense in the range of f. This result (for Δ) was recently proved in [39], but both results follow immediately from our first theorem (Section 1) together with a theorem of Bary [4, p. 635] according to which $\mathcal{H} \circ$ CBV consists of those continuous functions for which (*) is met. (See also [28].)

II. MONOTONICITY

During the last few years, a plethora of theorems, each concluding that a function is monotonic, has appeared in the literature. This is due in part to the fact that a number of authors have become interested in various sorts of generalized derivatives, and each such generalized derivative can give rise to a variety of monotonicity theorems. Many of these theorems follow a certain format which we examine in Section 1, below. Then in Section 2 we discuss a few results of a more abstract nature.

1. A Format for Monotonicity Theorems.

Many of the theorems which generalize the elementary theorem that a function F whose derivative is positive on an interval I_0 is increasing on I_0 follow a certain format. One assumes that F meets some regularity condition (e.g., $f \in C$, $f \in C_{ap}$ or $f \in \mathcal{DB}_1$), that some sort of generalized derivative exists except, perhaps, on some set which is small (e.g., countable) and that this generalized derivative is nonnegative except on some (possibly larger) small set. For the elementary theorem, the regularity condition is differentiability, and both small sets are empty. Often, a new monotonicity theorem improves an older one in that one or more of the hypotheses of the older theorem are weakened a bit. Although the weakening of a part of a hypothesis may appear minor, this weakening could require an entirely different sort of proof based on very delicate arguments.

We shall consider only one chain of theorems as an illustration of the preceding remarks. The interested reader may wish to consult the original articles appearing in this chain to get a real grasp of the work involved in establishing such a chain.

Our starting point is the following theorem of Goldowski and Tonelli established in 1928 and 1930. See Saks [66] for a proof.

Theorem (Goldowski and Tonelli). Let F satisfy the following conditions on I_0.

(i) F is continuous.

(ii) F' exists (finite or infinite), except perhaps on a denumerable set.

(iii) $F' \geq 0$ a.e.

Then F is nondecreasing on I_0.

In 1939, Tolstoff [69] obtained the following improvement of this theorem:

Theorem [Tolstoff]. Let F satisfy the following conditions on I_0:

(i) F is approximately continuous.

(ii) F'_{ap} exists (finite or infinite) except perhaps on a denumerable set.

(iii) $F'_{ap} \geq 0$ a.e.

Then F is continuous and nondecreasing on I_0.

Another generalization of the Goldowski-Tonelli theorem was obtained by Zahorski [76] in 1950.

Theorem [Zahorski]. Let F satisfy the following conditions on I_0.

(i) F has the Darboux property.

(ii) F' exists (finite or infinite) except perhaps on a denumerable set.

(iii) $F' \geq 0$ a.e.

Then F is continuous and nondecreasing on I_0.

It is now natural to ask the question: "Can one take the weaker of each pair of hypotheses in the last two theorems, and still infer monotonicity of F?"

This question has a negative answer [62], but if one assumes also that F is in the first class of Baire, the question has a positive answer. (Observe that the hypotheses of Zahorski's theorem imply F is in the first class of Baire.) The following theorem was established in [8][68].

Theorem. Let F satisfy the following conditions on I_0.

(i) F has the Darboux property and is in the first class of Baire.

(ii) F'_{ap} exists (finite or infinite) except, perhaps, on a denumerable set.

(iii) $F'_{ap} \geq 0$ a.e.

Then F is continuous and nondecreasing in I_0.

Technically speaking, what we loosely called a "chain" of theorems is not really a chain since neither Zahorski's theorem nor Tolstoff's generalizes the other.

Let us focus, for a moment, on the structure of the four theorems we stated and on the kinds of generalization frequently in the recent literature.

Condition (i) is a regularity condition. One can replace it with some other such condition, (e.g., symmetric continuity, preponderant continuity, qualitative continuity, etc.). It is then natural to deal with an appropriate generalized derivative (e.g. the symmetric derivative, preponderant derivative or qualitative derivative). Now, in each of our theorems the exceptional set of generalized nondifferentiability was taken to be at most denumerable, and the set on which the generalized derivative was not known to be nonnegative was taken to be a null set. And this is typical of many such theorems. What other possibilities are there for these exceptional sets? The negative of the Cantor function shows that care must be taken in attempting to replace the denumerable exceptional sets in condition (ii) by sets in some σ-ideal including nondenumerable sets, and it is clear that a σ-ideal containing sets of positive measure would very likely not work as exceptional sets for condition (iii). Roughly speaking, the denumerability of the exceptional sets in (ii) may stem from the principle that denumerable sets cannot influence growth patterns of continuous functions very much. (We state this principle vaguely because we know of no precise formulation of it - but denumerably many exceptions are often allowed in the hypotheses of theorems involving growth of continuous functions.) While the hypotheses of our four theorems did not always require continuity, the General Reduction Theorem of Section 2 below indicates that various other hypotheses can be reduced to the continuous case. On the other hand, the set on which generalized derivative fails to exist is usually a Borel set. Nondenumerable Borel sets always contain perfect sets of measure zero. Usually a Cantor-like function can then be constructed as a counterexample.

Regarding the exceptional sets in (iii) being null sets, we remark that if a set E has positive measure, there will always be a differentiable function F such that $\emptyset \neq \{F' < 0\} \subset E$. Thus, the null sets form a natural σ-ideal of exceptional sets in (iii).

But there are other possibilities obtained by controlling the growth of the function on the exceptional sets. For example, here is a theorem with a very simple proof.

<u>Theorem</u>. Let F satisfy the following conditions in I_0.

(i) F is continuous.

(ii) F(A) has measure zero, where $A = \{F' \text{ does not exist}\}$.

(iii) F(B) has measure zero, where $B = \{F' < 0\}$.

Then F is nondecreasing.

This theorem differs from the others we stated in that we are concerned with the images of the exceptional sets rather than the sets themselves. Many monotonicity theorems are of this sort, but they usually are stated in terms of the images of the sets where some extreme derivate (rather than the derivative) is not positive. For example, if the set on which $D^{+}F \leq 0$ maps onto a set with empty interior and F is continuous, then F is nondecreasing. (The same simple proof works.) We cannot change the hypothesis to $\{D^{+}F \leq 0\}$ has measure zero, however, (consider the negative of the Cantor function), but we would be able to make this change if we also assumed that $D^{+}F$ is in the first class of **Baire** [44]. We mentioned this now in anticipation of one of the open problems which we discuss in Chapter V.

We close this section by mentioning that a number of monotonicity theorems involving the growth of a function on an exceptional set can be found in Saks [66], Leonard [44] and Redheffer [64]. A relatively up-to-date bibliography of papers dealing with monotonicity theorems involving a variety of generalized derivatives can be found in [7].

2. Reduction Theorems and Abstract Theorems.

The theorems in Section 1 were all specific in nature: each involved a specific (perhaps generalized) derivative. We chose the few theorems that we did because they offered some sort of perspective on the way in which increasingly more general theorems developed historically. But the large number of choices one has for regularity conditions, for generalized derivatives and for exceptional small sets makes it clear that the total number of "reasonable" monotonicity conjectures is enormous. How can one transform the resulting chaos into order? Or, phrased differently, how can one get a hold on the distinction between a theorem and a plausible, but false, conjecture?

These questions are, of course, difficult to answer - they are not even well posed. But some recent results are of such a nature as to make it possible to obtain a whole family of theorems from a single theorem. The four results we shall discuss are of two types: "reduction" theorems which allow one to obtain monotonicity theorems about a large class of functions from analogous theorems about a smaller class; and "abstract" theorems, in which the definition of the generalized derivative is abstract rather than specific. (Our discussion will clarify what this means.) For example, an immediate application of our first reduction theorem gives Zahorski's theorem from the Goldowski-Tonelli theorem and an immediate application of our first abstract theorem is a monotonicity theorem in terms of extreme approximate derived numbers.

We mention that our statement about chaos and order is a serious one. So many monotonicity theorems have been proved in the last few years that it is really difficult to sort them out. And a catalogue of theorems and counter-examples would have only limited value. What is needed are real insights about what distinguishes a theorem from a false conjecture.

Let us turn to our first reduction theorem. Roughly speaking, it asserts that any monotonicity theorem valid for the class of continuous functions of bounded variation is also valid for the (much) larger class of Darboux functions in Baire class 1 ($\mathcal{DB}_1$). (A mild side condition is necessary here.) Thus, if one knows a function is in $\mathcal{DB}_1$ and satisfies some conditions (probably involving some generalized derivative being nonnegative on some large set) and wonders whether this condition implies that the function is increasing, we can (roughly speaking) assume the function is continuous and even of bounded variation and check whether the condition suffices under that stronger hypothesis.

We need a bit of notation. Let I_0 be a fixed interval and $\mathcal{P}$ a family of functions defined on I_0. For each interval $I \subset I_0$, let $\mathcal{P}(I)$ denote the restrictions of the functions in $\mathcal{P}$ to the interval I. Let $\mathcal{M}$ denote the family of non-decreasing functions on I_0, let $\mathcal{V}$ denote the family of functions of bounded variation on I_0 and let $\mathcal{B}$ denote the functions which are VBG on I_0. As usual, C will denote the continuous functions on I_0.

General Reduction Theorem [8]. If,

(i) $C\mathcal{V}\mathcal{P}(I) \subset \mathcal{M}(I)$ for every $I \subset I_0$ and

(ii) $\mathcal{D}\mathcal{B}_1\mathcal{P}(I) \subset \mathcal{B}(I)$ for every $I \subset I_0$, then

$$\mathcal{D}\mathcal{B}_1\mathcal{P}(I) \subset C\mathcal{M}(I) \quad \text{for every } I \subset I_0.$$

In essence, this theorem tells us that if a property (given by the family $\mathcal{P}$), is sufficiently strong to guarantee that a continuous function of bounded variation possessing that property must be nondecreasing, and that a Darboux-Baire 1 function possessing that property is VBG, then the property is also sufficiently strong to imply that each function in $\mathcal{DB}_1$ that possesses the property is continuous and nondecreasing. The theorem does not mention derivatives or generalized derivatives explicitly, but a typical such property will usually involve some generalized derivative. For example, if $\mathcal{P}$ denotes the family of functions F such that F' exists (finite or infinite) except perhaps on a denumerable set, and F' $\geq$ 0 a.e., then our Reduction Theorem allows us to infer Zahorski's theorem from the Goldowski-Tonelli theorem (In this case $\mathcal{P} \subset \mathcal{B}$. This is often true when the class $\mathcal{P}$ is given in terms of generalized derivatives.)

Using the Reduction Theorem, Leonard [44] has obtained a number of monotonicity theorems involving various sorts of generalized derivatives, (e.g., preponderant derivatives, qualitative derivatives, unilateral derivatives, and Dini derivatives). In addition, Bullen and Mukhopadhyay [18] and O'Malley [57] have applied the theorem to obtain monotonicity criteria in terms of Peano derivatives and selective derivatives respectively. Some of these, as well as other applications of the Reduction Theorem are discussed in [7].

A more specific reduction theorem was recently proved by O'Malley and Weil [59]. It is more specific because it deals only with the approximate derivative.

<u>Specific Reduction Theorem</u>. If

$$\Delta \mathcal{P}(I) \subset \mathcal{N}(I) \quad \text{for all } I \subset I_0$$

then

$$\Delta_{ap} \mathcal{P}(I) \subset \mathcal{N}(I) \quad \text{for all } I \subset I_0 .$$

Thus, a property sufficiently strong to imply monotonicity for differentiable functions is also sufficiently strong to imply monotonicity for approximately differentiable functions. It is easy to verify that if the condition is given in terms of properties of the ordinary derivative, one could also state it in terms of approximate derivatives. For example, the fact that a differentiable function with a nonnegative derivative a.e. is nondecreasing, implies that the same conclusion is valid for an approximately differentiable function whose approximate derivative is nonnegative a.e.

The O'Malley-Weil theorem requires <u>everywhere</u> differentiability. It would be of interest to know how much weakening of that requirement is possible. It would also be of interest to know what other generalized derivatives admit similar reduction theorems. Perhaps there exists a very general reduction theorem which could be applied to a number of generalized derivatives. If so, a great deal of order could be restored to the present chaotic state of affairs.

Let us turn now to a discussion of what we called abstract monotonicity theorems. The generalized derivatives which appear in the literature usually involve some sort of difference quotient approaching a limit in some sense. One such generalized derivative differs from another in the manner in which x is to approach x_0 in the calculations of the limit. Thus, the approximate derivative requires approach through a set having density 1 at x_0; the preponderant derivative weakens that requirement to a "preponderance of density" and the qualitative derivative replaces the notion of density with that of category.

Two recently-studied notions of derivative indicate the method of approach in more general terms. O'Malley [57] defined a notion of selective derivative as follows: From each interval $I \subset R$, select a point P_I from the interior of I. The collection of points obtained in this way is called a selection S. For $x_0 \in R$ and a given selection S, we define the selective derivative sF' of the function F at x_0 as

$$sF'(x) = \lim_{h \to 0} \frac{F(P_{[x_0, x_0+h]}) - F(x_0)}{P_{[x_0, x_0+h]} - x_0}$$

if this limit exists. (The notation $[x_0, x_0+h]$ denotes the interval determined by the points x_0 and x_0+h even if $h < 0$.)

Using the General Reduction Theorem, O'Malley obtained the following result.

O'Malley's Abstract Monotonicity Theorem. Let S be a selection and let F satisfy the following conditions on I_0:

(i) $F \in \mathcal{L}\mathcal{B}_1$

(ii) sF' exists except, perhaps, on some denumerable set

(iii) $sF' \geq 0$ a.e.

Then F is continuous and nondecreasing on I_0.

We consider O'Malley's theorem an abstract one because judicious choices of selections can give rise to monotonicity theorems involving specific generalized derivatives. For example, if F has an approximate derivative (possibly infinite) at each point of I_0, then there is a selection S such that $sF' = F'_{ap}$ on I_0. Thus each approximate derivative can be realized as a selective derivative. Thus, we can infer from O'Malley's theorem that an approximately differentiable function whose approximate derivative is nonnegative a.e. must be continuous and nondecreasing. This result has been known for some time, of course, but another of O'Malley's monotonicity theorems has as an immediate corollary a monotonicity theorem in terms of extreme approximate derivates which was new at the time: If F is measurable, $\underline{F}'_{ap} \geq 0$ a.e. and $\underline{F}'_{ap} > -\infty$ everywhere, then F is nondecreasing [57].

It is now natural to ask which of the other generalized derivatives can be realized as selective derivatives (even almost everywhere). Positive results in this direction could lead to new monotonicity theorems, and could also lead to a better understanding of what makes monotonicity theorems work.

We discuss very briefly another abstract monotonicity theorem due to Mastalerz-Wawrzyńczak [50]. Here one assigns to each $x \in I_0$ a family $T(x)$ of subsets of I_0 meeting certain natural conditions. Each set in $T(x)$ is called a T-neighborhood of x. The notion of a T-limit of a function F at a point x is also defined in a natural manner, as is the T-derivative, F'_T. If, for example, $T(x)$ is the family of sets containing x for which x is a density point, then $F'_T(x) = F'_{ap}(x)$. The main theorem asserts that if F and T satisfy certain conditions and $F'_T \geq 0$ a.e. on I_0, then F is nondecreasing and continuous on I_0. A special case of this abstract theorem is Zahorski's theorem of Section 1.

The theorem could possibly be useful for obtaining certain kinds of monotonicity theorems. Such theorems would be of a special nature, however, because the conclusion involves not only the monotonicity and continuity of the function, but also that the function be differentiable except, perhaps, on a denumerable set. Note, however, that this is also true of the theorems we discussed in Section 1, because a monotonic function is differentiable at each point of approximate differentiabilty. And even when differentiability except, perhaps, on a denumerable set, is not a conclusion of a monotonicity theorem, continuity usually is. This is so because the regularity part of the hypotheses of many monotonicity theorems includes the Darboux property, and a monotonic Darboux function must be continuous. Theorems for which the conclusion is simply that the function be monotonic do exist, of course. Some such theorems actually provide characterizations of monotonic functions. See, for example, Saks [66] for such characterizations with Zygmund's condition ($\overline{\lim}_{h\to 0^+} F(x-h) \leq F(x) \leq \overline{\lim}_{h\to 0^+} F(x+h)$ for all x) for regularity and Lee [43] with the regularity condition involving a notion of semi-absolute continuity.

III. DIFFERENTIATION OF TYPICAL CONTINUOUS FUNCTIONS

The first example of a continuous nowhere differentiable function is widely assumed to be due to Weierstrass (about 1875), although there appears to be some evidence that Bolzano had constructed such a function somewhat earlier. It was not until 1931 that the existence of such function was proved by use of the Baire Category theorem (see Banach [3] and Mazurkiewicz [54]). Shortly thereafter, Marcinkiewicz [46] and Jarnik [36][37][38] used the Baire Category theorem to show that typical continuous functions exhibit a great deal of pathology with respect to differentiation properties. (Here, and throughout this chapter, we shall use the term "typical continuous function" to mean that the set of functions which exhibit the property we are discussing is residual in the complete metric space $C = C[0,1]$.) More recently, other authors have obtained a number of similar results, each showing that the typical continuous function behaves pathologically with respect to differentiation and/or generalized differentiation.

In Section 1, we discuss some of the pathological behavior of typical continuous functions. Then, in Section 2, we present a result which indicates that the behavior, while pathological, is also very "regular" in nature.

1. Pathology.

Our starting point is the nowhere differentiability of a typical continuous function. Now, to say that F is nowhere differentiable is to say that at no point does F have a finite (two-sided) derivative. What happens if we allow derivatives to be infinite? Or if we allow derivatives to be one-sided? An inspection of the standard category argument shows that one can, by perhaps modifying the arguments a bit, allow either of these relaxations in the definition of the derivative and still conclude that a typical continuous function is nowhere differentiable (in the relaxed sense). But one cannot allow both relaxations simultaneously without losing the result! In fact, Saks [65] showed that a typical continuous function has an infinite unilateral derivative on a nondenumerable set. It was not until 1925 that the existence of a continuous function with no finite or infinite unilateral derivative at any point was proved. Besicovich constructed such a function in [5].

What happens if one replaces the ordinary derivative with some generalized derivative? Does one still get nowhere (generalized) differentiability of typical continuous functions? The answer depends, of course, on the specific generalized derivative one considers. Thus, Jarnik [38] has shown that a typical continuous function is nowhere approximately differentiable, Kostryko [41] has obtained the analogous result for symmetric differentiation, and Evans [23] for approximate symmetric differentiation. In the other direction, Marcinkiewicz [46], Jarnik [37] and Scholz [67] proved that typical continuous functions are differentiable almost everywhere in certain generalized senses. We discuss Scholz' result, the strongest of the three.

Suppose, first, that E is a set having the origin as a point of density and F is a function for which $\lim_{\substack{h \to 0 \\ h \in E}} \frac{F(x+h)-F(x)}{h}$ exists for some x. This limit is then the approximate derivative of F at the point x. If the limit exists for almost every x, then F is approximately differentiable a.e. in some strong sense because the same set E is involved for a.e. x.

Let us weaken the requirement by asking only that the <u>upper</u> density of E at the origin is 1, i.e., that

$$\overline{\lim_{h \to 0}}\, h^{-1} \mu(E \cap [0,h]) = 1.$$

If $\lim_{\substack{h \to 0 \\ h \in E}} \frac{F(x+h)-F(x)}{h}$ exists for some x, we say F is E-differentiable at x. Now the typical continuous F is nowhere approximately differentiable, but there always will be a set E with unit upper density at the origin such that F is E-differentiable a.e. There is a problem here, however. How do we know that different choices of E won't lead to entirely different E-derivatives? We don't! A typical F can have many different E-derivatives. Not only "can" - it does! How many? Scholz proved the following remarkable theorem.

<u>Theorem</u>. Let F be a typical continuous function and let f be an arbitrary measurable function. Then there exists a set E having unit upper density at the origin such that f is the E-derivative of F for almost every x.

Thus, not only is a typical continuous function a.e. differentiable in Scholz' sense, but we are free to pick in advance what its derivative is to be!

By taking constant values for f in Scholz' theorem, one sees immediately that a typical continuous F has every real number as an essential derived number a.e. This is Jarnik's theorem [37]. Jarnik also proved in [36] that every extended real number is an ordinary derived number of F at <u>every</u> point. These results give some sort of indication of how badly nondifferentiable a typical continuous function is.

2. Regularity.

The results of Scholz and Jarnik that we mentioned at the end of Section 1 suggest a great deal of pathology in the differentiability structure of typical continuous functions. But they also might suggest a regularity of sorts: two typical continuous functions behave very much alike. To obtain a clearer picture of this regularity, we shall discuss the manner in which the graphs of typical continuous functions intersect straight lines.

We mentioned in Chapter I that Gillis constructed a continuous function whose graph intersects each horizontal line in a perfect set. He claimed even more; namely that each nonvertical line intersects the graph in a perfect set. This latter claim was incorrect, however. In fact, Garg has observed that the graph of a continuous function must intersect many lines in sets containing isolated points. Suppose, for the moment, that Gillis' claim had been correct. Then the graph of Gillis' function would provide a "clear picture" of a continuous function for which each real number is a (bilateral) derived number at each point of (0,1).

Now Gillis' claim was not correct, but there are continous functions which behave very much like Gillis' was supposed to. In fact, the typical continuous function does! To formulate this typical behavior, we need a bit of terminology. Let θ be a direction, let F be in C and let L_θ denote the family of lines in the direction θ which intersect the graph of F. We say that F behaves <u>normally</u> in the direction θ if the graph of F intersects

(i) the two extreme lines of L_θ in singletons,

(ii) the lines of some denumerable "dense" subset of L_θ in the union of a nowhere dense perfect set with a singleton,

(iii) all other lines of L_θ in a nowhere dense perfect set.

<u>Theorem</u> [16]. The typical continuous function behaves normally except in a denumerable dense set of directions. In each exceptional direction, the behavior is normal except for a single line for which the intersection contains exactly two isolated points.

This theorem lends substance to the statement "typical functions look alike."

Because of the existence of isolated points in the intersections of the graph of F with certain lines, one cannot "see" the nondifferentiability of F as easily as one could from Gillis' claimed function. (These isolated points arise from relative strict extrema of functions of the form $F(x) - \gamma x$). But much of the nondifferentiability structure appears plausible from the picture, and, in fact, much of it follows from some of the preliminary theorems needed to prove the stated theorem. It would be of interest to obtain an improvement to the stated theorem - one from which many of the results we discussed in Section 1 could be visualized.

Finally, we mention that every continuous function F possesses a certain type of "internal" differentiability structure [14]. Each nonempty perfect set P contains a non-empty perfect subset Q such that $F|Q$ is differentiable. Furthermore for x outside some denumerable set, there exists a perfect set P containing x as a two-sided limit point such that $F|P$ is differentiable.

Let us call a set A a differentiable road for F provided $F|A$ is differentiable. The statements of the preceding paragraph indicate something about the kinds of differentiable roads possessed by <u>all</u> continuous functions. In the other direction, the typical continuous function possesses no differentiable road of positive measure and no dense differentiable road. This last fact can be contrasted with a theorem of Blumberg's which asserts that every function f possesses a dense continuity road (i.e. a dense set D such that $F|D$ is continuous).

There is a great deal known about nowhere differentiable functions - both typical and nontypical - that we have not been able to discuss in this short chapter. We refer the interested reader to a number of recent papers on the subject by K. M. Garg.

IV. THE ALGEBRA GENERATED BY Δ'

Consider the function $F(x) = x^2 \sin \frac{1}{x}$ $(F(0) = 0)$ with derivative $F'(x) = -\cos \frac{1}{x} + 2x \sin \frac{1}{x}$ $(F'(0) = 0)$. Simple considerations of the expression for F' show that the function $f(x) = \cos \frac{1}{x}$ $(f(0) = 0)$ must be a derivative. But one can also show that the function $f^2 = f \cdot f$ is not a derivative. This example shows that the product of two derivatives need not itself be a derivative. Two questions now arise:

(i) Under what circumstances is the product of two (or more) derivatives itself a derivative? and

(ii) How can one characterize the class of functions expressible as a product of two (or of n) derivatives?

The first of these questions has been studied extensively and the present state of knowledge has been summarized and discussed by Fleissner in the recent survey [24]. We have nothing to add to that survey.

The second question was posed by Solomon Marcus in 1977. While little work has been done on that question (and none of it published), it offers a number of interesting possibilities and it raises the larger question of characterizing Alg Δ', the algebra of functions generated by Δ'.

A bit is known regarding the second question. My two colleagues S. Agronsky and R. Biskner and I observed that the characteristic function of a closed set is always expressible as a product of two derivatives, but the characteristic function of a nonempty proper open subset of R can never be represented as a product of any number of derivatives. These results have been extended in a number of ways by J. Mařík, who also showed the existence of functions representable as a product of $n+1$ (but not of n) derivatives (for each $n = 2,3,\ldots$). His work in this connection leads to the following very instructive example. Let $G = R\setminus\{0\}$ and let f be a function continuous on G such that $1 \le f \le 2$ on G and such that the sets $\{f = 1\}$ and $\{f = 2\}$ have density 1/2 at 0. For each $\alpha \in R$ let $f_\alpha = f$ on G and $f_\alpha(0) = \alpha$. Then

(i) $f_\alpha \in \Delta'$ if and only if $\alpha = \frac{3}{2}$,

(ii) f_α is a product of no more than n derivatives if and only if $\alpha \ge ((1+2^{1/n})/2)$.

One can show that $\{((1+2^{1/n})/2)^n\}$ decreases to $\sqrt{2}$. Thus, for $\alpha \le \sqrt{2}$, f_α cannot be represented as a finite product of derivatives. As we lower the value of α from $\frac{3}{2}$ to $\sqrt{2}$, we require increasingly more derivatives for a product - representation of the otherwise well-behaved functions f_α, the representation disappearing entirely when $\alpha = \sqrt{2}$.

Let us complicate things a bit. What functions f can be expressed in the form $f = g' + h'k'$ $(g,h,k \in \Delta)$? No complete answer is known to this question, but some interesting results have been obtained by Mařík. For example, if f is locally bounded and for each $x \in R$ there exists a function φ_x continuous on $R - \{x\}$ such that $\lim_{t\to x} f(t) - \varphi_x(t) = 0$, then f can be represented in that form.

In particular, each function possessing finite unilateral limits at each point admits the representation. The same is valid for each function of the form $\sum \alpha_K \chi_{E_K}$, where the sets E_K are closed pairwise disjoint sets and the numbers α_K are nonnegative with $\sum \alpha_K < \infty$. The function

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ \frac{1}{q} \text{ if } x = \frac{p}{q} \text{ in lowest terms} \end{cases}$$, furnishes an example.

If a function f admits the representation $f = g' + h'k'$ it must, of course, be in $\mathcal{B}_1$. What else must be true? The answer to that question does not seem to be known. In fact we have no example of a Baire 1 function not representable in that form to offer! If there are such functions, then it becomes natural to ask whether $\mathcal{B}_1 = \text{Alg } \Delta'$. If the answer is affirmative, then one can establish without difficulty that there must be a positive integer N such that each $f \in \mathcal{B}_1$ can be expressed in terms of sums and products of no more than N derivatives. If the answer is negative, then there are many questions to ask. We mention that each $f \in \mathcal{B}_1$ is expressible as a uniform limit of functions in Alg Δ'. In fact each $f \in \mathcal{B}_1$ can be expressed as a uniform limit of functions of the form $g' + hk'$ $(g,h,k \in \Delta)$. Why of that particular form? We shall answer that question, but we prefer to postpone the answer until we have briefly discussed the class of functions f representable in the form $f = g' + hk'$ $(g,h,k \in \Delta)$, which seems to be of some interest. This class is discussed in [1] and denoted there by $[\Delta']$.

Suppose we know g, h and k are differentiable on R and $f = g' + hk'$. What does that say about f? If k' is summable then the product hk', and therefore f, must be a derivative. But, in general, $hk' \notin \Delta'$, so $f \notin \Delta'$. Nonetheless, f must possess some derivative-like structure. For example, there must exist a dense open set T and functions $F \in \Delta$ and G differentiable on T such that $F' = f$ on $R \setminus T$ and $G' = f$ on T. The existence of such T, F and G are also sufficient for f to be in $[\Delta']$. Several other characterizations of $[\Delta']$ can be found in [1].

One of the features of the class $[\Delta']$ which may make it useful is that it contains certain important classes of functions related to differentiation theory. Thus, each approximately continuous function, each approximate derivative, and each function in O'Malley's class $\mathcal{B}_1^*$ is in $[\Delta']$. For the approximate derivative heavy use is made of a property obtained by O'Malley [58].

Now the fact that $\mathcal{B}_1^* \subset [\Delta']$ implies immediately that a Baire 1 function with finite range is in $[\Delta']$ and this implies that $\mathcal{B}_1$ is contained in the uniform closure of $[\Delta']$.

The fact that $\Delta'_{ap} \subset [\Delta']$ may be useful to an understanding of why an approximate derivative possesses so many properties of a derivative. We discuss this question in Chapter V, below.

What role does the function g' play in the representation $f = g' + hk'$? Can we, for example, represent each function in $[\Delta']$ as hk' with $h,k \in \Delta$? The answer is negative. In fact even an approximately continuous approximate derivative with only one point of discontinuity cannot always be expressed in that form, or even as a product of two or more derivatives.

The central problem is probably that of characterizing Alg Δ'. But there are also many questions, some of which may be rather difficult, concerning the structures of functions which admit certain specific representations in terms of derivatives. Solutions to some of these problems could possibly require a deeper understanding of the structure of derivatives than we currently have. If so, the methods of a solution might be more important than the answers themselves.

V. OPEN PROBLEMS

We have already mentioned a number of open problems in differentiation theory. We now turn to discussions of three additional problems. (We shall not discuss the important problem of characterizing derivatives because that problem has been discussed extensively elsewhere [71]).

1. Derivatives Versus Approximate Derivatives.

Although the notion of approximate derivative is more general than that of (ordinary) derivative, each approximate derivative possesses many of the properties of derivatives. In fact, each of the many properties shown (since 1950) to be valid for ordinary derivatives has also been shown (since 1960) to be valid for approximate derivatives. The timing is not surprising: the Zahorski 1950 article [76] was the starting point for many of the investigations about derivatives, and the Goffman-Neugebauer 1960 article [32] was the starting point for many of the corresponding investigations about approximate derivatives. The chart below summarizes some of the results.

Property	Δ'	Δ'_{ap}
$\mathcal{M}_2$	Denjoy [22]	Marcus [48]
$\mathcal{M}_3$	Zahorski [76]	Weil [72]
$\mathcal{M}_4$	Zahorski [76]	*
$\mathcal{J}$	Weil [73]	Weil [73]
$\mathcal{M}^*$	Preiss [63]**	Preiss [63]
$\mathcal{M}_2^*$	Preiss [63]***	Preiss [63]
$\mathcal{M}_3^*$	Preiss [63]	Preiss [63]

---

*Zahorski showed that $b\Delta' \subset \mathcal{M}_4$. Since $b\Delta'_{ap} = b\Delta'$, the corresponding result $b\Delta'_{ap} \subset \mathcal{M}_4$ is trivially valid.

and * These results involve derivatives and approximate derivatives which are allowed to be infinite. Preiss characterized the associated sets for such functions and found no distinction between the classes of derivatives and approximate derivatives with respect to associated sets.

In addition, recent investigations [27], [59], have shown that much of the behavior of an approximate derivative F'_{ap} can be accounted for on the set on which $F'_{ap} = F'$.

Some of the similarities in behavior can be explained by the fact that an approximate derivative is also a selective derivative (see Chapter II), and some can be explained by the representation $F'_{ap} = g' + hk'$ (see Chapter IV). But there still are two problems which we believe should be addressed:

1) Find a more satisfying explanation for the similarities, ideally one which could apply equally well to explain why some other generalized derivatives behaves so much like ordinary derivatives while some others do not.

2) Find a single property of derivatives that is not also a property of approximate derivatives.

In connection with 2), we mention that the known differences between approximate derivatives and ordinary derivatives seem to involve the primitives, or, what amounts to the same thing, integration. For example, if $f = F'_{ap}$, then F need not be continuous and f need not be integrable in the Denjoy sense.

2. The Baire Class of Extreme Derivates of Continuous Functions.

Suppose a continuous function F has nonnegative extreme derivates a.e. Must F be nondecreasing? We saw in Chapter II that the negative of the Cantor function provides a negative answer to this question, but we also mentioned that if we had also required that $D^+F \in \mathcal{B}_1$, we would have obtained an affirmative answer to the question. This situation is typical in the sense that the assumption $D^+F \in \mathcal{B}_1$ often allows desirable conclusions which are not valid without that assumption. For example, Mukhopadhyay [56] has shown that the Dini derivates of a continuous function possess some Zahorski-like properties under certain conditions including membership in $\mathcal{B}_1$. In particular, if

(i) $F \in C$, (ii) $D^+F \in \mathcal{B}_1$, (iii) $D^-F \geq D^+F \geq D_-F$ and (iv) $-\infty < D^+F < \infty$ except perhaps for some denumerable set, then $D^+F \in \mathcal{M}_2$. It follows readily from this result that the derivative (possibly infinite) of a continuous function is in Zahorski's class $\mathcal{M}_2$.

The aforementioned theorems, and others like them, suggest the problem of determining conditions under which the Dini derivates of a continuous function are in $\mathcal{B}_1$. This problem seems to have been first posed by Solomon Marcus in 1960 [47]. Let's take a moment to discuss what is known.

a) If $D^+F \in \mathcal{B}_1$, then D^+F must be continuous on a residual set. This implies that F is differentiable on a residual set. The converses are not valid, as the negative of the Cantor function shows.

b) The Dini derivates of a continuous function are always in $\mathcal{B}_2$, although they can be very badly behaved. The function F - G of Chapter I, Section 1b, satisfies a Lipschitz condition but each of its Dini derivates takes on every value between -1 and +1 continuum many times on every subinterval of (0,1). It is possible to construct an absolutely continuous function F such that D^+F takes each rational value on a set having positive measure in each interval and each irrational value on a null set having continuum many points in each interval [9]. Thus, the fact that $D^+F \in \mathcal{B}_2$ is not very helpful in taming D^+F,

c) We cannot conclude that $D^+F \in \mathcal{B}_1$ from the knowledge that some other Dini derivate is in $\mathcal{B}_1$ [44].

d) Marcus [47] proved that a <u>necessary</u> condition for all Dini derivates to be in $\mathcal{B}_1$ is that the set of points of non-differentiability be the union of a null set of the first category with a nowhere dense set. Leonard [44] showed that a <u>sufficient</u> condition is that this set be a denumerable set of type G_δ.

e) R. Keston has given examples of continuous functions F and G such that $D^+F \in \mathcal{B}_1$, $D^+G \in \mathcal{B}_1$, but $D^+(F+G) \notin \mathcal{B}_1$. (He did not publish this result.)

f) The Dini derivates of a continuous function F are semi-Baire 1. Thus, for example $\{D^+F < c\} \in F_\sigma$ for each $c \in R$, but $\{D^+F > c\}$ need not be.

g) Results analogous to those in f) for functions F in Baire class α have recently been advanced by Misik [33]. Hájek [34] has shown that the extreme bilateral derivates of an <u>arbitrary</u> function are in $\mathcal{B}_2$. (One can ask for conditions under which these derivates are in $\mathcal{B}_1$.)

One reason the problem of finding conditions which are both necessary and sufficient for D^+F to be in $\mathcal{B}_1$ is difficult is that it is not immediately apparent what kind of conditions one should seek. The discussion in b), above, shows that standard regularity conditions on F do not offer much promise of being sufficient. And the fact that one Dini derivate can be in $\mathcal{B}_1$ without all four Dini derivates being in $\mathcal{B}_1$ suggests that conditions involving restricted differentiability are of limited value. (More precisely, conditions involving differentiability of the functions $F|P$, P perfect, cannot be both necessary and sufficient for a specific derivate to be in $\mathcal{B}_1$. It is conceivable that such conditions could characterize those functions <u>all</u> of whose derivates are in $\mathcal{B}_1$.)

3. Approximate Continuity of Derivatives.

It is easy to prove that the set of points of continuity of a derivative must be a dense set of type G_δ. One can also prove without much difficulty [7] that each such set must be the set of continuity of some derivative.

We pose the analogous problem of characterizing the set of points of approximate continuity of a derivative.

Not much is known about this set. The following remarks may lend some perspective to the problem.

a) A derivative, being measurable, must be approximately continuous a.e.

b) A derivative, being continuous on a residual set, must also be approximately continuous on a residual set.

c) Thus, the set of points of approximate discontinuity of a derivative must be a first-category null set. It seems likely that each null set of type F_σ is the set of points of approximate discontinuity of some derivative.

d) Let N be an arbitrary null set. Then χ_N is approximately continuous exactly on $\sim N$. (Thus, each null set is the set of approximate discontinuity of some measurable function.) Now the upper and lower essential boundaries of χ_N coincide everywhere, but χ_N does not agree with these boundaries on N. This situation cannot occur for a bounded derivative since $b\Delta' \subset \mathcal{M}_1$. One might, therefore, be able to obtain a necessary condition on the set of approximate continuity of a derivative through the use of essential boundaries. If this condition were also sufficient, we would very likely have a characterization valid for those functions whose values lie between the values of the essential boundaries at each point; i.e., for

$$\{f: \operatorname{ess} \varliminf_{x \to x_0} f(x) \leq f(x_0) \leq \operatorname{ess} \varlimsup_{x \to x_0} f(x)\} \quad \text{for all } x_0.$$

There are certain related questions one can ask. For example, the "typical" bounded derivative is approximately discontinuous on some dense set. (More precisely, if $b\Delta'$ is furnished with the sup norm, then $\{f \in b\Delta'$: f is approximately continuous on some interval$\}$ is a first category subset of $b\Delta'$.) The proof of this statement is not difficult [7]. How much stronger a statement is possible? Most likely, any pathology (with respect to approximate continuity) possible of a derivative will be typical of bounded derivatives.

We mention one more question because an answer to this question would have been useful on several occasions. . Suppose $f, g \in \Delta'$, f continuous exactly on A and g approximately continuous exactly on B. If $A \subset B$, does there exist $h \in \Delta'$ such that h is continuous exactly on A and approximately continuous exactly on B?

REFERENCES

1. S. Agronsky, R. Biskner, A. Bruckner and J. Mařík, Representations of functions by derivatives, to appear.

2. A. Baernstein and D. Waterman, Functions whose Fourier series converge uniformly for every change of variable, Indiana Univ. Math. J. 22 (1972), 569-576.

3. S. Banach, Über die Baire'sche Kategorie gewisser Funktionenmengen, Studia Math., 3 (1931), 174-179.

4. N. Bary, Memoire sur la representation finie des fonctions continue, Math. Ann. 103 (1930), 185-248 and 598-653.

5. A. Besicovitch, Diskussion der stetigen Funktionen im Zusammenhang mit der Frage über ihre Differentierbarkeit, Bull. Acad. Sci. de Russie, 19 (1925), 527-540.

6. J. Blažek, E. Borák and J. Malý, On Köpcke and Pompeiu functions, Čas. pro pěst. mat. 103 (1978), 53-61.

7. A. Bruckner, Differentiation of Real Functions, Springer, Berlin, 1978.

8. ____________, An affirmative answer to a problem of Zahorski and some consequences, Mich. Math. J., 13 (1966), 15-26.

9. ____________, Some remarks on extreme derivates, Canadian Math. Bull. 12 (1969), 385-388.

10. ____________, Density preserving homeomorphisms and a theorem of Maximoff, Quart. J. Math. Oxford 21 (1970), 337-347.

11. ____________, On transformations of derivatives, Proc. Amer. Math. Soc., 48 (1975), 101-107.

12. ____________, Inflexible derivatives, Quart. J. Math. Oxford, 29 (1978), 1-10.

13. ____________, Differentiability a.e. and approximate differentiability a.e., Proc. Amer. Math. Soc., 66 (1977), 294-298.

14. A. Bruckner, J. Ceder and M. Weiss, On the differentiability structure of real functions, Trans. Amer. Math. Soc. 142 (1969), 1-13.

15. A. Bruckner, R. Davies and C. Goffman, Transformations into Baire 1 functions, Proc. Amer. Math. Soc., 67 (1977), 62-66.

16. A. Bruckner and K. Garg, The level structure of a residual set of continuous functions, Trans. Amer. Math. Soc., 232 (1977), 307-321.

17. A. Bruckner and C. Goffman, Differentiability through change of variables, Proc. Amer. Math. Soc. 61 (1976), 235-241.

18. P. Bullen and S. Mukhopadhyay, On the Peano derivatives, Canadian J. Math. XXV (1973), 127-140.

19. M. Chaika and D. Waterman, On the invariance of certain classes of functions under composition. Proc. Amer. Math. Soc. 43 (1974), 345-348.

20. G. Choquet, Application des propriétés descritives de la fonction contingent à la théorie des fonctions de variable reélle et à la géometrie différentielle des variétés cartésiennes, J. Math. Pures Appl.,(19) 26 (1947), 115-226.

21. A. Denjoy, Sur les fonctions dérivées sommables, Bull. Soc. Math. France, 43 (1915), 161-248.

22. ___________, Sur une propriété des fonctions dérivées, Enseignement Math., 18 (1916), 320-328.

23. M. Evans, On continuous functions and the approximate symmetric derivatives, Coll. Math. 31 (1974), 129-136.

24. R. Fleissner, Multiplication and the Fundamental Theorem of Calculus: A survey, Real Analysis Exchange, 2 (1976), 7-34.

25. R. Fleissner and J. Foran, A note on differentiable functions, Proc. Amer. Math. Soc. 69 (1978), 56.

26. ___________, Transformations of differentiable functions, Coll. Math., to appear.

27. R. Fleissner and R. O'Malley, Conditions implying the summability of approximate derivatives, to appear.

28. J. Foran, Continuous Functions: A survey, Real Analysis Exchange, 2 (1977), 35-104.

29. ___________, Changes of variable which preserve almost everywhere approximate differentiability, to appear.

30. J. Gillis, Note on a conjecture of Erdos, Quart. J. Math. Oxford 10 (1939), 151-154.

31. C. Goffman, Everywhere differentiable functions and the density topology, Proc. Amer. Math. Soc. 51 (1975), 250.

32. C. Goffman and C. Neugebauer, On approximate derivatives, Proc. Amer. Math. Soc. 11 (1960), 962-966.

33. C. Goffman and D. Waterman, A characterization of the class of functions whose Fourier series converge for every change of variable, J. London Math. Soc. 10 (1975), 69-74.

34. O. Hájek, Note sur la mesurabilité B de la dérivée supérieure, Fund. Math., 44 (1957), 238-240.

35. P. Hancock, Homeomorphic transformations of approximately continuous functions into derivatives, doctoral dissertation, UCSB, 1979.

36. V. Jarník, Über die Differenzierbarkeit stetiger Funktionen, Fund. Math. 21 (1933), 48-58.

37. ___________, Sur les nombres dérivées approximatifs, Fund. Math. 22 (1934), 4-16.

38. ___________, Sur la dérivabilité des fonctions continues, Spisy Privodov, Fak. Univ. Karlovy, 129 (1934), 3-9.

39. L. Kaplan and S. Slobodnik, Monotone transformations and differential properties of functions, Mat. Zam. 22 (1977), 859-871. (In Russian.)

40. Y. Katznelson and K. Stromberg, Everywhere differentiable, nowhere monotone functions, Amer. Math. Monthly 81 (1974), 349-354.

41. P. Kostryko, On the symmetric derivative, Coll. Math. 25 (1972), 265-267.

42. M. Laczkovich and G. Petruska, On the transformers of derivatives, Fund. Math. 100 (1978), 179-199.

43. C. Lee, An analogue of the theorem of Hake-Alexandroff-Looman, Fund. Math. 100 (1978), 69-74.

44. J. Leonard, Some conditions implying the monotonicity of a real function, Rev. Roum. Math. Pures Appl. 12 (1972), 757-780.

45. J. Lipiński, Mesure et dérivee, Colloq. Math., 8 (1961), 83-88.

46. J. Marcinkiewicz, Sur les nombres dérivés, Fund. Math., 24 (1935), 304-308.

47. S. Marcus, Sur certains problèmes et théorèmes concernant la continuité et la dérivabilité des fonctions, Monat. Math. 64 (1960), 119-130.

48. ______________, On a theorem of Denjoy and on approximate derivatives, Monatsh. Math. 66 (1962), 435-440.

49. ______________, Sur les dérivées dont les zeros forment un ensemble frontière partout dense, Rend. Circ. Mat. Palermo (2) 12 (1963), 1-36.

50. M. Mastalerz-Wawrzyńczak, On a certain condition of the monotonicity of functions, Fund. Math 97 (1977), 197-198.

51. I. Maximoff, On continuous transformation of some functions into an ordinary derivative, Ann. Scuola Norm. Sup. Pisa., 12 (1943), 147-160.

52. ______________, Sur la transformation continue de fonctions, Bull. Soc. Phys. Math. Kazan, (3) 12 (1940), 9-41. (Russian; French summary.)

53. ______________, Sur la transformation continue de quelques fonctions en dérivées exactes, Bull. Soc. Phys. Math. Kazan, (3) 12 (1940), 57-81. (Russian; French summary.)

54. S. Mazurkiewicz, Sur les fonctions non-derivables, Studia Math., 3 (1931), 92-94.

55. L. Misik, Halbborelsche Funktionen und Extreme Ableitungen Math. Slov. 27 (1977), 409-421.

56. S. Mukhopadhyay, On Dini derivatives and on certain classes of Zahorski, Coll. Math. 25 (1972), 105-111.

57. R. O'Malley, Selective derivatives, Acta Math. Acad. Sci. Hung., 29 (1977), 77-97.

58. ______________, Decomposition of approximate derivatives, Proc. Amer. Math. Soc. 69 (1978), 243-247.

59. R. O'Malley and C. Weil, The oscillatory behavior of certain derivatives, Trans. Amer. Math. Soc., 234 (1977), 467-481.

60. G. Petruska and M. Laczkovich, Baire 1 functions, approximately continuous functions, and derivatives, Acta Math. Acad. Sci. Hung., 25 (1974), 189-212.

61. D. Pompeiu, Sur les fonctions dérivées, Math. Ann., 63 (1906), 326-332.

62. D. Preiss, Approximate derivatives and Baire classes, Czech. Math. J. 21 (1971), 373-382.

63. ______________, Level sets of derivatives, to appear.

64. R. Redheffer, Increasing functions, to appear.

65. S. Saks, On the functions of Besicovitch in the space of continuous functions, Fund. Math., 19 (1932), 211-219.

66. ___________, Theory of the integral, Monografie Mathematyczne 7, Warszawa-Lwow, 1937.

67. J. Scholz, Essential derivations of functions in C[a,b], to appear.

68. T. Swiatkowski, On the conditions of monotonicity of functions, Fund. Math., 59 (1966), 189-201.

69. G. Tolstoff, Sur quelques proprietes des fonctions approximativement continues, Rec. Math. (Mat. Sbornik) N.S., 5 (1939), 637-645.

70. ___________, Parametricheskoe differentsirovaniei Uzkii Integral Denjoy, Math. Sbornik 53 (1961), 387-392.

71. D. Waterman, On functions of bounded deviation, Acta. Sci. Math. 36 (1974), 259-263.

72. C. Weil, On properties of derivatives, Trans. Amer. Math. Soc. 114 (1965), 363-376.

73. ___________, A property for certain derivatives, Indiana Univ. Math J. 23 (1973/74), 527-536.

74. ___________, On nowhere monotone functions, Proc. Amer. Math. Soc. 56 (1976), 388-389.

75. W. Wilkosz, Some properties of derivative functions, Fund. Math., 2 (1921), 145-154.

76. Z. Zahorski, Sur la première dérivée, Trans. Math. Soc., 69 (1950), 1-54.

University of California
Santa Barbara