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A Note on Λ-Bounded Variation

Let $\Lambda = \{\lambda_n\}$ be a nondecreasing sequence of real numbers such that $\lambda_n \to \infty$ and $\Sigma 1/\lambda_n = \infty$. A function f on an interval I is said to be of Λ bounded variation ($f \in \Lambda BV$) if there exists an M such that

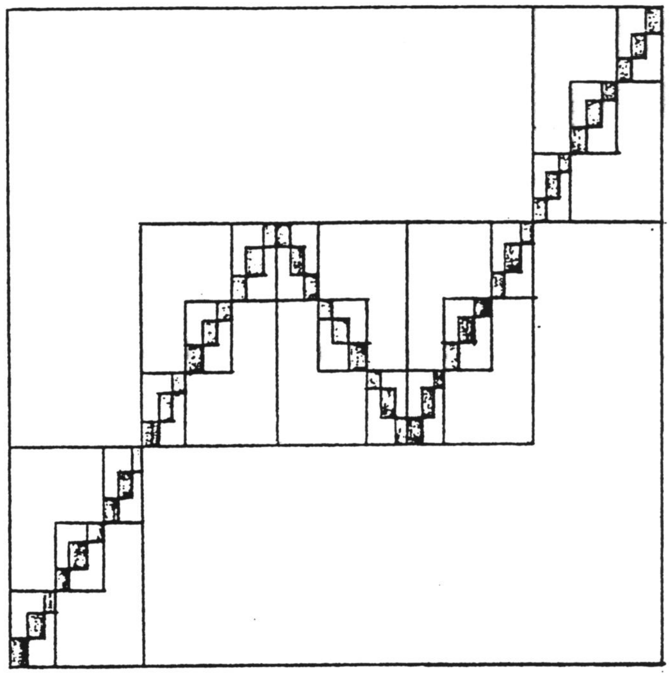
$$\Sigma |f(b_n)-f(a_n)|/\lambda_n < M$$

for each finite collection of non-overlapping intervals (a_n,b_n) contained in I. The supremum of such sums is called the (total) Λ variation of f on I which we abbreviate as $\Lambda(f)$. If $\lambda_n = n$, ΛBV is called harmonic bounded variation (HBV). For a discussion of the properties of functions of HBV and ΛBV and their connection with Fourier Series, see [1,2,3,4,5].

For $\Lambda = \{\lambda_i\}$, let $\Lambda_m = \{\lambda_m,\lambda_{m+1},\ldots\}$. Since Λ is nondecreasing, it is easily seen that if $f \in \Lambda BV$, the sequence $\Lambda_m(f)$ is nonincreasing. Professor Daniel Waterman [5] posed the following problem. If $f \in \Lambda BV$, does $\Lambda_m(f) \to 0$ as $m \to \infty$? The purpose of this note is to show that $\Lambda_m f$ need not tend to 0. For convenience we shall write w_i in place of $1/\lambda_i$.

Example. There is a continuous function f and a sequence $\{w_i\}$ with $w_i \to 0$ and $\Sigma w_i = \infty$ such that for each m, $\frac{4}{5} \le \Lambda_m(f) < \infty$.

Construction: Call the following operation D(n) upon a rectangle R: Divide the rectangle into 15 smaller equal rectangles by dividing it by 5ths in the x-direction and by 3rds in the y-direction. Let A represent the three rectangles central to R along with two rectangles chosen in opposing corners of R. Divide each rectangle of A into 9 smaller congruent rectangles, by 3rds in the x- and y-directions. Let A_1 be the central subrectangles of rectangles of A along with the two opposing corner subrectangles from each rectangle in A chosen in such a fashion that the points in the rectangles of A_1 form a connected set. Form A_i from the rectangles of A_{i-1} in the same fashion that A_1 was formed from A. Continue this until i=2n. The collections A, $A_1, \ldots, A_{2n}$ of subrectangles will be referred to as the stages of the operation D(n) with A being the first stage and A_{2n} the last stage. Thus the operation D(1) performed on the unit square looks like the figure on the next page (the shaded portions in the figure represent the rectangles of A_2). Note that the operation D(n) determines $5 \cdot 3^{2n}$ subrectangles within a given rectangle.



We define f as follows: Perform the operation D(1) on the unit square E_o in the fashion illustrated. On each of the rectangles of the set $E_1=A_2$ which result, perform the operation D(2) in such a fashion that the points in the resulting collection E_2 of rectangles form a connected set. Similarly form a collection of rectangles E_n by performing the operation D(n) on the rectangles in the set E_{n-1} in such a way that the points in the rectangles of E_n form a connected set. If we denote by F_n the collection of points in the rectangles of E_n, then $\cap F_n$ is the graph of a function f and since this intersection is closed, f is continuous. Note that the rectangles involved in the stages of D(n) have heights 3^{-k} where $n^2 \leq k < (n+1)^2$.

In what follows, when we refer simply to the operation $D(n)$, we mean the operation $D(n)$ upon E_{n-1}.

We define Λ as follows: Let $w_1 = 1$. Set $N(n) = 5^n 3^{n(n-1)}$, which for $n \geq 1$ is the number of rectangles in the first stage of $D(n)$; and note that

$$(*) \qquad N(n) - N(n-1) \geq 4 \cdot 5^{n-1} 3^{n^2-n}.$$

If $N(n-1) < j \leq N(n)$, set $w_j = (3/5)^n$ (these w_j will be called the n-th block of Λ). Then $w_j \to 0$, and by (*) the sum of the n-th block is at least $(4/5)3^{n^2}$; hence, $\Sigma w_i = \infty$.

Now the first stage of $D(n)$ yields a partition of the domain of f into $N(n)$ subintervals on each of which the oscillation of f is 3^{-n^2}. Since the n-th block of Λ contains $N(n) - N(n-1)$ terms each equal to $(3/5)^n$, this partition shows that

$$\Lambda_{N(n-1)}(f) > [N(n) - N(n-1)](3/5)^n 3^{-n^2},$$

which together with (*) gives $\Lambda_{N(n-1)}(f) > 4/5$.

In order to determine that $\Lambda(f) < \infty$, let $\{[a_i,b_i]\}$ be any finite collection of non-overlapping subintervals of $[0,1]$. Since f is continuous, we may assume it takes on its maximum and minimum values on $[a_i,b_i]$ precisely at the endpoints of this interval (otherwise, choose a subinterval of the given interval). Let R_i denote the largest subrectangle involved with the construction of the sets F_n (i.e., R_i is one of the rectangles chosen at some stage of some operation $D(n)$) such that $R_i \subset [a_i,b_i] \times [0,1]$. Then the points $(a_i,f(a_i))$ and $(b_i,f(b_i))$ lie in a single rectangle of the previous stage or they lie in two adjacent rectangles from the previous stage. Since at each stage new rectangles were obtained by dividing the previous heights in 3rds, $|f(a_i) - f(b_i)|$ cannot exceed 6 times the height of R_i. Thus if we confine our attention to intervals whose endpoints are the abscissas of corners of the rectangles of the various stages, the estimate of $\Lambda(f)$ will be off by at most a factor of 6. Furthermore, if $[c_i,d_i] = pr_x(R_i)$, then since $\{w_i\}$ is non-increasing, it is easily shown that $\Sigma|f(c_i) - f(d_i)|w_i$ assumes its maximum value if the oscillations in the sum are arranged in descending order and we shall assume that they are.

Case 1. It is first shown that those rectangles R_i which are associated with $w_j = (3/5)^n$ and which were obtained at some point after the operation $D(n)$ was performed are a small part of the sum. In fact, if we were to allow them every available place in the series they would contribute to the sum no more than

$$1\cdot(1/3)\cdot 1 + 5(1/3)^4(3/5) + 5^2 3^2(1/3)^9(3/5)^2 + \dots + 5^n 3^{n^2-n}(1/3)^{(n+1)^2}(3/5)^n + \dots$$

which is less than $\Sigma(1/3)^n = 1/2$.

Case 2. We now consider the remaining rectangles. Fix $n \geq 1$ and let $R^1, R^2, \ldots, R^t$ be the rectangles that both come from one of the stages of $D(n)$ and are associated with $(3/5)^j$ for some $j \geq n$. (Case 1 takes care of those otherwise associated.) Then, letting Σ_n denote the part of the sum that involves the rectangles R^i and letting $h(R^i)$ denote the height of R^i, we easily see that

$$\Sigma_n \leq (3/5)^n \sum_{i=1}^{t} h(R^i).$$

But, since $h(R^i)$ equals the sum of the heights of all the rectangles from the last stage of $D(n)$ that are contained in R^i, we have

$$(**) \quad \Sigma_n \leq (3/5)^n Q_n (1/3)^{(n+1)^2-1} = Q_n (1/3)^{n(n+1)} (1/5)^n ,$$

where Q_n is the total number of rectangles from the last stage of $D(n)$ that are contained in some R^i $(i=1,2,\ldots,t)$. Furthermore, since the width of each such rectangle is $(1/3)^{n(n+1)}(1/5)^n$, the right side of (**) is just the measure of the x-projection of the union of the R^i, which we denote by M_n. Consequently, the rectangles considered in this case contribute to the sum an amount less than the sum of the M_n, which is less than or equal to 1.

Finally, by combining the estimates of Cases 1 and 2 and taking into account the factor of 6, we have $\Lambda(f) < 6(1 + 1/2)$.

Problem. *If* $f \in HBV$, *does* $\lim_{m\to+\infty} \Lambda_m(f) = 0$ *where*

$\Lambda_m = \{\frac{1}{m+i}\}_{i=0}^{\infty}$.

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