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Note on the Oscillatory Behavior of Certain Derivatives

Recently, R. J. O'Malley and C. E. Weil [8] obtained an interesting property for both the approximate derivative and the Peano derivative. They have encouraged me to look at the problem of whether the approximate Peano derivative has the same property. The purpose of this note is to give an affirmative answer to the question. In fact, for each positive integer k, we have the following result.

Theorem A_k. If the k^{th} approximate Peano derivative, $f_{(k)}$, of a real-valued function f exists, is finite and attains both values M and -M on an interval, then there exists an open subinterval on which $f_{(k)}$ is just the ordinary iterated k^{th} derivative, $f^{(k)}$, of the function f and also attains both the values M and -M.

Theorem A_1 is just theorem 4.1 in [8] for the approximate derivative, and if the approximate Peano derivative is replaced by the ordinary Peano derivative in Theorem A_k for $k \geq 2$, the result is just theorem 4.2 in [8] for the Peano derivative. We note that separate proofs were given in [8] for their theorem 4.1 and theorem 4.2. The proof there for theorem 4.1 is much shorter than that for theorem 4.2, mainly due to the application of lemma 3.3 there, which is a result concerning some structure properties of approximately continuous functions. But we do not know how to obtain a result analogous to this lemma for $(k-1)^{th}$ Peano derivatives (cf. remark 1 at the end), so that whether it is possible to use the argument used in the proof of theorem 4.1 to give a shorter proof of theorem 4.2 in [8] is still an open question.

Although the interesting argument used in the proof of theorem 4.2 is rather tedious, we realize that it can be adopted to give a proof for theorem A_k here, even for $k = 1$. To see this, let us show that approximate Peano derivatives have all the properties of Peano derivatives which are used in the proof of theorem 4.2 in [8]. We list these as a sequence of lemmas and indicate references when the results are known. (At this point, it seems worthwhile to mention that it is not always true that approximate Peano derivatives possess whatever properties that ordinary Peano derivatives have. See, for example, [6].)

<u>Lemma 1</u> (see [1] or [3]). If the finite k^{th} approximate Peano derivative, $f_{(k)}$, of a function f is bounded below or above on an interval [a,b], then $f_{(k)}$ is just the ordinary iterated k^{th} derivative of the function f there.

Lemma 2 (see [1] and [2]). Every exact finite k^{th} approximate Peano derivative on an interval is of Baire class one and possesses the Darboux property on the interval.

Lemma 3 (see [4]). If the finite k^{th} approximate Peano derivative, $f_{(k)}$, of a function f is Lebesgue integrable on an interval $[a,b]$, then

$$f_{(k-1)}(b) - f_{(k-1)}(a) = \int_a^b f_{(k)}(x)dx,$$

where $f_{(k-1)}$ is the $(k-1)^{th}$ approximate Peano derivative of f when $k \geq 2$, and $f_{(k-1)} = f$ when $k = 1$.

Lemma 4. Suppose that the k^{th} approximate Peano derivative at the point x, $f_{(k)}(x)$, of a function f exists and is finite, and suppose that the $(k-1)^{th}$ approximate Peano derivative, $f_{(k-1)}$, of f exists and is finite on an open interval containing x. Then for each positive number ε there exists a positive number δ such that for any number y with $0 < |y-x| < \delta/2$ there are y_1 and y_2 with $y_1 < y < y_2$ satisfying

(1) $|y_i - y| < \varepsilon|y - x|$ for $i = 1, 2$; and

(2) $|f_{(k-1)}(z) - f_{(k-1)}(x) - (z - x) f_{(k)}(x)| < \varepsilon|z - x|$ for z in a set of positive measure contained in $[y_1,y]$, and in a set of positive measure contained in $[y,y_2]$.

We remark that lemma 4 is a unified generalization of lemma 3.2 and lemma 3.5 listed in [8], noting that in the generalization the assumption is weaker while the conclusion is stronger. Thanks are due to J. Mařík since this lemma follows easily from his theorem 3 in [7]. To see this, taking $\eta = \epsilon/2$ and $j = k - 1$ in theorem 3 there, then one sees easily that for each y with $0 < |y - x| < \delta/2$ there always exist y_1, y_2 with $|y_i - x| < \delta$ and $y_1 < y < y_2$ such that (1) in lemma 4 holds and also $|y_i - y| > \frac{\epsilon}{2} d(x, L_i)$ where $L_1 = [y_1, y]$ and $L_2 = [y, y_2]$, so that (2) in lemma 4 must hold, too.

With lemmas 1 to 4, theorem A_k can be proved following the line given in the proof of theorem 4.2 in [8]. We omit all the detail here. Instead, we make some remarks to conclude the note.

<u>Remark 1</u>. In certain non-absolutely convergent integration theories the concept of generalized absolute continuity plays an important role. Lemma 3.3 listed in [8] mentioned previously is closely related to the following result (cf. [4]): If the approximate derivative of a function exists and is finite everywhere in a closed interval, then the function is [ACG] on the interval, i.e., the interval can be decomposed into countably many closed sets on each of which the function is absolute continuous. There is a similar result for the Peano derivative (cf. [9]): If the k^{th} Peano derivative of a function exists and is finite everywhere in a closed

interval, then the $(k-1)^{th}$ Peano derivative of the function is [ACG] on the interval. We hope that this result may give light to the question on how to obtain a result for Peano derivatives which is analogous to lemma 3.3 listed in [8]. Note that we have mentioned previously that it is of interest to have such an analogue, which, as C. E. Weil sees it, will also be useful in the decomposition of Peano derivatives into ordinary derivatives.

Remark 2. On the other hand, we do not know (cf. [5]) whether the $(k-1)^{th}$ approximate Peano derivative of a function is [ACG] on an interval when the k^{th} approximate Peano derivative of the function exists and is finite on the interval $(k \geq 2)$. The technique used in the proof of theorem 4.2 in [8] might be useful to attack the problem.

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