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An Integral Inequality

During the course of investigating a certain class of Plateau problems, the first author came upon the following fact about real functions:

If f, a non-negative Lebesque measurable function on $[0,\infty)$, and $p \geq 0$ are such that $\int_0^T t^{-p}f(t)dt = \infty$ for all $T > 0$ then for each $q > p$ there is a sequence $t_n > 0$ converging to zero with $t_n f(t_n) \leq q \int_0^{t_n} f(t)dt$.

This statement follows from the elementary theorem below.

Theorem. Suppose $f:[0,\infty) \to [0,\infty)$ is a measurable function and $0 \leq p < q$. If $\int_0^T t^{-p}f(t)dt = \infty$ for each $T > 0$ then the set $(0,B) \cap \{t:tf(t) \leq q \int_0^t f(s)ds\}$ has positive measure for each $B > 0$.

Proof. We prove the contrapositive statement. Suppose $B > 0$ is such that

$$(0,B) \cap \{t:tf(t) > q \int_0^t f(s)ds\} \qquad (*)$$

has measure B. For the convenience of exposition, we denote by F the function $F(t) = \int_0^t f(s)ds$, $t \in (0,B)$.

Then (*) yields F is positive, absolutely continuous and increasing on $[a,b]$, where $[a,b]$ is any closed subinterval of $(0,B)$. Since the logarithm function is Lipschitzian on the interval $F([a,b])$, the composition $\log \circ F$ is also absolutely continuous on $[a,b]$. The function G on $[a,b]$ given by

$$G(t) = \log(t^{-q}F(t)) = (\log \circ F)(t) - q \log t$$

is absolutely continuous. G is increasing because, for almost every t in $[a,b]$,

$$\frac{dG}{dt}(t) = \frac{f(t)}{\int_0^t f(s)ds} - \frac{q}{t}$$

is positive due to (*) . The end result is that the function $t^{-q}F(t)$, $t \in (0,B)$, is increasing. Let $0 < T < B$ and denote by M the constant $T^{-q}F(T)$. Then $\int_0^t f(s)ds = F(t) \le Mt^q$, $0 \le t \le T$. For $n \ge 0$ and $0 < R < 1$, we have

$$\int_{TR^{n+1}}^{TR^n} t^{-p}f(t)dt \le (TR^{n+1})^{-p} \int_{TR^{n+1}}^{TR^n} f(t)dt$$

$$\le MT^{q-p}R^{-p}R^{(q-p)n} .$$

Consequently,

$$\int_0^T t^{-p}f(t)dt \le MT^{q-p}R^{-p} \sum_{n=0}^{\infty} R^{(q-p)n} .$$

The right-hand side is finite because $p < q$. The theorem is proved.

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