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A GENERALIZATION OF THE SCHROEDER-BERNSTEIN THEOREM[1]

The following theorem has a corollary the well-known Schroeder-Bernstein theorem [1,p.20].

Theorem. Let A and B be sets, F map subsets of A to subsets of B and G map subsets of B to subsets of A. If for every nested sequence $\{A_i\}$ of subsets of A

$$F(\cap A_i) = \cap F(A_i),$$

and if for every sequence $\{B_i\}$ of subsets of B

$$G(\cup B_i) = \cup G(B_i),$$

then there exist subsets α of A and β of B such that α and $G(\beta)$ partition A and β and $F(\alpha)$ partition B.

Proof. Let $\alpha_1 = A$, $\beta_j = B\setminus F(\alpha_j)$, $\alpha_{j+1} = A\setminus(\bigcup_{i=1}^{j} G(\beta_i))$, for $j = 1,2,\ldots$, and $\alpha = \bigcap_{j=1}^{\infty} \alpha_j$. One must then define β to be $B\setminus F(\alpha)$ and prove that $G(\beta) = A\setminus\alpha$.

Since F commutes with intersection on nested sequence of subsets of A it follows that $F(\alpha) = \bigcap_{j=1}^{\infty} F(\alpha_j)$. Taking complements with respect to B on both sides of this equality yields $\beta = \bigcup_{j=1}^{\infty} \beta_j$. Thus,

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$$G(\beta) = \bigcup_{j=1}^{\infty} G(\beta_j)$$

$$= \bigcup_{j=1}^{\infty} \left(\bigcup_{i=1}^{j} G(\beta_i) \right)$$

$$= \bigcup_{j=1}^{\infty} (A \setminus \alpha_{j+1})$$

$$= \bigcup_{j=1}^{\infty} (A \setminus \alpha_j)$$

$$= A \setminus \alpha,$$

proving the theorem.

For any function h and any subset S of the domain of h let $h_*(S)$ denote the set $\{h(s): s \in S\}$.

Corollary 1. Let f map A into B and g map B into A. Suppose that $f^{-1}\{(b)\}$ is a finite set for each point b of B. Then there exist subsets α and β of A and B respectively such that α and $g_*(\beta)$ partition A and β and $f_*(\alpha)$ partition B.

Proof. Apply the theorem with $F = f_*$ and $G = g_*$. It must be shown that if $\{A_i\}$ is a nested sequence of subsets of A then $f_*(\cap A_i) \supseteq \cap f_*(A_i)$. Let $s \in f_*(A_i)$ and let $a_1, a_2, \ldots, a_n$ be the points of A having image s under f. It suffices to show that $a_j \in \cap A_i$ for some j, $1 \leq j \leq n$. Assume the contrary. Since $A_1 \supseteq A_2 \supseteq \ldots$ there exist positive integers $m(j)$, $1 \leq j \leq n$, such that $a_j \notin A_i$ for $i \geq m(j)$. Let $M = \max \{m(1), m(2), \ldots, m(n)\}$. Then $a_j \notin A_M$ for $1 \leq j \leq n$ which contradicts $s \in f_*(A_i)$, proving the corollary.

Corollary 2. (Schroeder - Bernstein) If there exists a 1-1 function f from set A into set B and a 1-1 function g from B into A, then there exists a 1-1 function from A onto B.

Proof. By Corollary 1 there exist subsets α and β of A and B respectively such that α and $g_*(\beta)$ partition A and β and $f_*(\alpha)$ partition. Define h by

$$
\begin{aligned}
h(a) &= f(a) \text{ if } a \in \alpha \\
&= g^{-1}(a) \text{ if } a \in g_*(\beta).
\end{aligned}
$$

Then h is a 1-1 function from A onto B.

REFERENCES

1. E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer-Verlag, New York, 1969.

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