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Classes of Continuous Real Functions

by

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Suppose that T is a topology in [0,1] different than the natural topology. Let C([0,1],T) and C([0,1]) denote the class of all real functions defined on [0,1] which are continuous with respect to T and to the natural topology, respectively (in both cases we assume that the range of functions is endowed with the natural topology).

Sometimes it can happen that C([0,1],T) = C([0,1]). For example this equality holds if T is a topology in which for every x_0 and every neighborhood U of x_0 there exists a number $\delta > 0$ such that $U \cap (x_0-\delta, x_0+\delta) \cap [0,1]$ is residual in $(x_0-\delta, x_0+\delta) \cap [0,1]$ (the so-called qualitative topology).

Observe that if T is not stronger than the natural topology (recall that this entails that these two topologies are different), then there exists an open set A, which does not belong to T and C([0,1],T) $\neq$ C([0,1]). Indeed, the function f(x) = x is not continuous with respect to T.

Hence, if we want to characterize all topologies for which the above equality holds, it is natural to suppose that T is stronger than the natural topology. A similar question can be put in a more general setting, namely, if X is an arbitrary non-empty set, T_0,T are two topologies on X and we consider only real functions with the range endowed with the natural topology. However, we shall be mainly interested with the case when $T_o \subset T$ and (X,T_o) is a compact space. Let $C(X,T)$ and $C(X,T_o)$ denote respectively the classes of all real T-continuous and T_o-continuous functions defined on X. Also, we shall use the terminology, for example, T- (or T_o-) neighborhood, T- (or T_o-) accumulation point, T- (or T_o-) limit to make a distinction between two topologies under consideration.

This survey will consist of two parts. In the first we shall discuss the general case; in the second we shall present some results connected with the special case when $X = [0,1]$, T_o - the natural topology. The topology terminology used in this paper is consistent with that found in Engelking [1].

1. Suppose that (X,T_o) is a compact space and that $T \supset T_o$. Obviously $C(X,T_o) \subset C(X,T)$. Also every function $f \in C(X,T_o)$ is bounded. Thus, if $C(X,T_o) = C(X,T)$, then every function $f \in C(X,T)$ is bounded.

This necessary condition is also sufficient in the case when (X,T_o) is a metrizable space.

Indeed, suppose that f is a T-continuous function which is not T_o-continuous. Let x_o be a point of T_o-discontinuity of f. Then there exists an $\varepsilon_o > 0$ such that for every T_o-neighborhood U of x_o there is a point

$x \in U$ for which

$$|f(x) - f(x_o)| > \varepsilon_o .$$

Put

$$f_1(x) = \frac{f(x) - f(x_o)}{\varepsilon_o} \frac{\pi}{2} ,$$

$$f_2(x) = \min \left\{ f_1(x), \frac{\pi}{2[1+\varsigma^2(x,x_o)]} \right\}$$

and

$$f_3(x) = \max \left\{ f_2(x), \frac{-\pi}{2[1+\varsigma^2(x,x_o)]} \right\},$$

where ς denotes a metric yielding the topology T.

Then f_3 is a T-continuous function, $|f_3(x)| < \frac{\pi}{2}$ for every $x \in X$ and $\sup_{x \in X} |f_3(x)| = \frac{\pi}{2}$.

If we put now

$$f_4(x) = \mathrm{tg}\, f_3(x) \doteq \tan f_3(x),$$

then we obtain a T-continuous function which is unbounded - a contradiction.

We shall say that a topological space (X,T) is called a *-compact space if for an arbitrary pair of families of sets $\{F_y\}$ and $\{G_y\}$, where $y \geq y_o$, which satisfy the conditions:

1^o F_y is T-closed, G_y is T-open for $y \geq y_o$

2^o $y_o \leq y_1 < y_2 \Rightarrow F_{y_1} \supset G_{y_1} \supset F_{y_2} \supset G_{y_2}$

3^o $F_y \neq \emptyset \neq G_y$ for $y \geq y_o$

the following condition is satisfied

4^o $\bigcap_{y \geq y_o} F_y = \bigcap_{y \geq y_o} G_y \neq \emptyset$

It can be proved without special difficulty that every real T-continuous function is bounded if and only if the space (X,T) is *-compact (Kocela [2]). Indeed, suppose that (X,T) is a *-compact space and f is a real T-continuous unbounded function. Then the pair of families of sets $F_y = \{x: |f(x)| \geq y\}$ and $G_y = \{x: |f(x)| > y\}$ for $y \geq 0$ satisfies conditions 1^o-3^o and $\bigcap_{y \geq y_o} F_y = \bigcap_{y \geq y_o} G_y = \emptyset$ because of the finiteness of f - a contradiction.

Suppose now that (X,T) is not *-compact. Then there exists a pair of sets $\{F_y\}$, $\{G_y\}$, $y \geq y_o$, fulfilling 1^o-3^o such that $\bigcap_{y \geq y_o} F_y = \bigcap_{y \geq y_o} G_y = \emptyset$. Obviously we can assume $G_{y_o} = F_{y_o} = X$. If we define the function f in the following way:

$$f(x) = \sup \{y: x \in F_y\} \text{ for } x \in X,$$

then it is easy to see that f is unbounded and, after a bit of reflection, that f is T-continuous.

From the above it follows immediately that if (X,T_o) is a compact metrizable space, then $C(X,T) = C(X,T_o)$ if and only if (X,T) is *-compact. Also it is possible to prove (Nonas [7]) that a metrizable space (X,T) is compact if and only if it is *-compact.

So if (X,T_o) is a compact metrizable space and $T \supset T_o$, then the following conditions are equivalent:

(i) $C(X,T_o) = C(X,T)$

(ii) $C(X,T) \subset B(X)$,

where B(X) denotes the class of all bounded functions defined on X.

Simultaneously (ii) is equivalent to the *-compactness of (X,T). One can ask whether the metrizability is essential. The answer is included in the following theorems (Nonas [7]).

If (X,T_o) is a compact space (even countably compact will suffice) and if there exists a point $x_o \in X$ such that x_o is a T_o-accumulation point of X but x_o is not a T_o-accumulation point of any countable subset of X, then there exists a topology $T \supset T_o$ such that (X,T) is *-compact and $C(X,T_o) \neq C(X,T)$. It suffices to take for T the coarsest topology which contains both T_o and $\{x_o\}$. As the example of a compact topological space (X,T_o) in which there exists a point x_o having the above mentioned properties we can take a set of denumerable ordinals together with the smallest nondenumerable ordinal Ω endowed with the topology generated by sets of the form $\{0\} \cup \{z: y < z \leq x\}$, where $y < x \leq \Omega$.

The above theorem of Nonas shows that compactness of (X,T_o) without metrizability is not sufficient for the equivalence of conditions (i) and (ii) (or (i) and *-compactness, of course) for some topology $T \supset T_o$.

However, it is possible to weaken the assumption concerning the metrizability. In [7] it is proved that if (X,T_o) is a compact space such that every one-point set in X is a T_0-G_δ set and if T is a topology finer than T_o, then the conditions (i) and (ii) are equivalent.

Moreover, it is sufficient to assume only pseudocompactness instead of compactness of (X,T_o) (a topological space (X,T_o) is called pseudocompact if it is a Tychonoff space and every continuous real-valued function defined on X is bounded, compare [1]). The proof is essentially the same as in the case of metrizability.

The last theorem is stronger than the previous, because there exist compact spaces in which every one-point set is of type G_δ and which are not metrizable (Urysohn [12], pp. 936-939).

On the other hand, it is possible that (X,T_o) is a compact space, T is some topology such that $T \supset T_o$ and (i) is equivalent to (ii), but there exists a point $x_o \in X$ which is not a T_0-G_δ set. One can construct an example in the following way (Nonas [7]):

Let X be an uncountable set and let $x_o \in X$. Let $E \in T_o$ if and only if $E \subset X$ and $x_o \notin E$ or $E \subset X$, $x_o \in E$ and $\mathrm{card}(X-E) < \aleph_o$. It is easy to see that (X,T_o) is a compact space and $\{x_o\}$ is not a T_o-G_δ set. It is a little more difficult is to show that the only topology T finer than T_o for which (X,T) is *-compact is T_o itself. So (i) is obviously equivalent to (ii) for $T=T_o$ and we are done.

From the above it follows that if (X,T_o) is a compact space, then the supposition that every one-point set is a

T_0-G_δ set is sufficient, but not necessary for the equivalence of (i) and (ii) for a topology $T \supset T_0$.

However, if we assume that the topology T_0 is determined by order, then this supposition becomes necessary and sufficient (Nonas [7]).

Question 1. *How can the class of compact spaces* (X,T_0) *for which (i) is equivalent to (ii) be characterized?*

To show how close the notion of *-compactness is to notions of compactness, pseudocompactness and countable compactness we shall quote several theorems in which functionally T-open set means a set of the form $f^{-1}(G)$, where f is a real T-continuous function defined on X and G is open subset of the real line.

The *-compactness of the space (X,T) is equivalent to each of the following conditions ([4]):

a. Every locally finite functionally T-open cover of X is finite

b. For every decreasing sequence $\{G_n\}$ of nonempty functionally T-open subsets of X the intersection of their T-closures is nonempty.

If we assume that (X,T) is a normal space, then *-compactness is equivalent to countable compactness ([6]).

Here is another simple characterization of *-compactness in terms of continuous images ([6]): a topological space (X,T) is $*$-compact if and only if for every topological space (Y,T_1) and for every continuous transformation $f\colon X\to Y$ the space $(f(X),T_1|f(X))$ is $*$-compact.

There is also a close connection between the $*$-compactness of the topological space (X,T) and the behavior of sequences of T-continuous real functions defined on X. Namely, a topological space (X,T) is $*$-compact if and only if every sequence $\{f_n\}$ of T-continuous real functions defined on X which is locally uniformly convergent on X to a function f converges to f uniformly on X ([5]). Recall that the sequence $\{f_n\}$ converges locally uniformly to f if every point $x_0 \in X$ has the neighborhood $U(x_0)$ in which the convergence is uniform.

The notion of $*$-compactness allows the possibility of generalization of the classical Dini theorem. This generalization is stated in [10] in the following form: A topological space (X,T) is $*$-compact if and only if every monotone sequence $\{f_n\}$ of real T-continuous functions which is convergent on X to a T-continuous real function f converges to f uniformly on X.

Question 2. *Is it possible to prove theorems similar to the last theorems for generalized (Moore-Smith) sequences of T-continuous functions?*

To finish this section we present some connections between $*$-compactness and the approximation theorem of

Stone and Weierstrass. We shall say that a topological space (X,T) is a Weierstrass-Stone [W-S] space if and only if the following condition is fulfilled: for every family $\mathcal{F} \subset C(X,T)$ which includes all constant functions, separates points of X, and is closed under taking linear combinations, the uniform closure of $\mathcal{F}$ is equal to $C(X,T)$ (that is, if in (X,T) the thesis of the Stone-Weierstrass theorem holds).

The following theorems have been proved in [11]: If (X,T) is a W-S space, then (X,T) is *-compact. If (X,T_o) is a compact metrizable space and T is a topology on X finer than T_o, then (X,T) is a W-S space if and only if it is a *-compact space.

2. Suppose now that $X = [0,1]$ and T_o is the natural topology on the unit segment. Since (X,T_o) is compact and metrizable, from the first section we have that if T is a topology on [0,1] finer than the natural topology, then $C([0,1],T) = C([0,1])$ if and only if $([0,1],T)$ is a *-compact space.

However, in the case of functions of a real variable we are able to present a more detailed discussion. Throughout this section T will always denote a topology in [0,1] finer than the natural topology in [0,1].

The following conditions were introduced in [9] during a study of asymmetry of functions:

(W) If $x_n \in (E_n)'_T$ for $n=1,2,\ldots$ (where $(E_n)'_T$ is the set of all T-accumulation points of E_n) and x_n converges to x (in the natural topology), then

$$x \in \left(\bigcup_{n=1}^{\infty} E_n\right)'_T$$

(W') For an arbitrary point $x \in [0,1]$ and for arbitrary T-neighborhood U of x there exists a number $\delta > 0$ such that the set $\{(x-\delta, x+\delta) - U\}'_T$ is empty.

In [9] it was proved that (W) is equivalent to (W'). Kocela in [2] has proved that if (X,T) fulfills (W), then $C([0,1],T) = C([0,1])$. For suppose that f is a T-continuous function which is not continuous at some point x_o. Then there exists an $\varepsilon > 0$ and a sequence $\{x_n\}$ converging to x_o such that for every n we have the inequality, say, $f(x_n) < f(x_o) - \varepsilon$. From the T-continuity we have for each n a T-open set E_n such that $x_n \in E_n$ (so obviously $x_n \in (E_n)'_T$) and $f(x) < f(x_o) - \varepsilon$ for $x \in E_n$. But then $x_o \in \left(\bigcup_{n=1}^{\infty} E_n\right)'_T$ from the assumption, so $f(x_o)$ cannot be the T-limit of f at x_o - a contradiction.

The inverse implication is false. The example of a topology T finer than a natural topology for which (W) does not hold but the classes $C([0,1],T)$ and $C([0,1])$ are equal, can be constructed in the following way:

The set $U \in T$ if and only if it has density one at each of its points (right-hand density at 0 and left-hand density at 1, of course) and if for every $x_o \in U$ there exists a number $\delta > 0$ such that

$$(x_o-\delta, x_o+\delta) \cap (Q+x_o) \cap [0,1] = U \cap (Q+x_o) \cap [0,1],$$

where Q is a set of rational numbers and $Q+x_o = \{x+x_o : x \in Q\}$ ([3]). Every T-continuous function is approximately continuous, so is a Baire 1 function. The proof that this function is continuous depends on the fact that every T-neighborhood of any point x is dense in some ordinary neighborhood of x (see the discussion on the following pages).

To prove that T does not fulfill the condition (W) it suffices to observe that if for every n

$$E_n = [2^{-n}-4^{-n}, 2^{-n}] - \bigcup_{w_k \in Q} (w_k-\epsilon_k^{(n)}, w_k+\epsilon_k^{(n)}),$$

where $\{\epsilon_k^{(n)}\}$ is a sequence of positive numbers such that $\sum_{k=1}^{\infty} \epsilon_k^{(n)} < 4^{-n}$ and $x_n \in (E_n)'_T$ (such a point does exist), then x_n converges to zero, but 0 is not a T-accumulation point of $\bigcup_{n=1}^{\infty} E_n$.

Notice that the condition (W) also appeared in the generalization of a theorem on monotonicity in [8]

However, it is possible to modify the condition (W) to obtain a condition which is equivalent to

$C([0,1],T) = C([0,1])$. This condition, unfortunately, is not very useful, because it depends on the notion of functionally open sets.

Namely, the following condition:

(W_1) If $x_n \in E_n$ and E_n is functionally T-open for $n=1,2,\ldots$, and x_n converges to x (in the natural topology), then

$$x \in \left(\bigcup_{n=1}^{\infty} E_n\right)'_T$$

This condition is a necesary and sufficient condition for the equality $C([0,1],T) = C([0,1])$, ([3]).

The proof that (W_1) is sufficient is exactly the same as for the condition (W). If the condition (W_1) is not fulfilled, then it is possible to choose a sequence $\{U_n\}$ of disjoint functionally T-open sets and a sequence $\{x_n\}$ of points convergent to x_o such that $x_n \in U_n$ and $x_o \in \left(\bigcup_{n=1}^{\infty} U_n\right)'_T$. Next we can find a sequence $\{f_n\}$ of T-continuous non-negative functions such that $U_n = \{x: f_n(x)>0\}$. Multiplying, if necessary, each f_n by a suitable constant we can assume that $\sup f_n \geq n$. Then the function $f = \sum_{n=1}^{\infty} f_n$ is T-continuous and unbounded, so $C([0,1],T) \neq C([0,1])$.

It is easy to observe that if $C([0,1],T) = C([0,1])$, then every interval $[a,b] \subset [0,1]$ is a T-connected set. In [2] it was proved that this equality implies that every T-neighborhood of an arbitrary point $x_o \in [0,1]$

is a dense set (with respect to the natural topology) in some interval $(x_0-\delta, x_0+\delta) \cap [0,1]$.

Suppose that it is not the case. It means that there exists a point x_0, a T-neighborhood U of x_0, and a sequence of disjoint intervals $\{(a_n,b_n)\}$ such that $U \cap \bigcup_{n=1}^{\infty}(a_n,b_n) = \emptyset$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x_0$.

Let the function f be defined in the following way:

$$f(x) = \begin{cases} 0 & \text{for} \quad x \notin \bigcup_{n=1}^{\infty}(a_n,b_n) \\ 1 & \text{for} \quad x=x_n=\frac{a_n+b_n}{2},\ n=1,2,\ldots \\ \text{linear in every interval } [a_n,x_n] & \\ \qquad\qquad \text{and} \quad [x_n,b_n]\ . & \end{cases}$$

This function is T-continuous, but not continuous at x_0 - a contradiction.

It is natural to ask if these two conditions are sufficient for the equality $C([0,1],T) = C([0,1])$. The answer is negative ([13]).

Using the transfinite induction one can prove that there exists a function $F: [0,1] \to (0,1)$ such that:

1. For every $y \in (0,1)$ the set $F^{-1}\{y\}$ is dense in [0,1] (with the natural topology)
2. For every interval $[a,b] \subset [0,1]$ and for every set $P \subset [a,b] \cap [0,1]$ which is closed in $[a,b] \times [0,1]$ and such that Proj P (i.e. the

projection of P on the x-axis) includes some interval $[c,d] \subset [a,b]$, there exists a point $x' \in [a,b]$ such that $(x',F(x')) \in P$.

If T is the coarsest topology in $[0,1]$ such that $C([0,1]) \subset C([0,1],T)$ and $F \in C([0,1],T)$, then T is the required topology. The inequality $C([0,1],T) \neq C([0,1])$ is obvious. From the first property of the function F it follows immediately that every T-neighborhood of any x_0 is dense in some neighborhood of x_0, because the basis for T is the class of all sets of the form $F^{-1}(C) \cap D$, where C and D are open sets in the natural topology. The proof of the fact that every interval $[a,b] \subset [0,1]$ is a T-connected set is more delicate. It is based on the second property of F and on the fact that for every pair G_1,G_2 of open sets in the plane and for every straight line P included in the plane either there is a point on the line P which does not belong to the union of orthogonal projections $\hat{G}_1,\hat{G}_2$ of these two sets on this line or there is a linear segment on P which is included in $\hat{G}_1 \cap \hat{G}_2$.

However, in [3] the following theorem was proved: If every T-neighborhood of each point $x_0 \in [0,1]$ is dense (in the natural topology) on some interval, then every T-continuous real function having a dense set of points of continuity is continuous (the last two words denote continuity with respect to the natural topology).

The proof goes along the following line: If x_o is a point of discontinuity of f, then there exist an $\varepsilon > 0$ and an interval [a,b] containing x_o such that both of the sets $\{x: |f(x) - f(x_o)| < \varepsilon/3\}$ and $\{x: |f(x) - f(x_o)| \geq 2\varepsilon/3\}$ are dense in [a,b]. So f cannot have a point of continuity in this interval - a contradiction.

In particular, under this assumption every T-continuous Baire 1 function is continuous.

Hence in the last counterexample (from [13]) it is impossible to obtain a topology T such that C([0,1],T) - C([0,1]) includes the Baire 1 functions. It is possible to modify the construction in such way that the function F is Lebesgue measurable, so C([0,1],T) - C([0,1]) does include the Lebesgue measurable functions.

Question 3. *Is it possible to construct such a topology T which fulfills both conditions (i.e. every interval [a,b]   [0,1] is a T-connected set and every T-neighborhood U of each point x_o is dense in some interval $(x_o-\delta, x_o+\delta)$ [0,1]) such that C([0,1],T) - C([0,1]) is nonempty and*

a) includes some Baire function?

b) includes some Baire α function (for a given ordinal α)?

c) is included in the set of all Baire functions ?

d) is included in the class of Baire α *functions (for a given ordinal* α*) ?*

e) is included in the class of Lebesgue measurable functions ?

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