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Set Theoretic, Measure Theoretic, Combinatorial, and Number Theoretic Problems Concerning Point Sets in Euclidean Space

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In this paper I will discuss a few problems which are on the borderline of geometry, number theory, and set theory and which have occupied me for a long time. Perhaps they are more difficult than important, but I find them fascinating.

It is a common paradox that problems on infinite sets are often very much simpler than problems on finite sets. This prompted Ulam and myself to state the following somewhat impudent and unfortunately somewhat inaccurate motto:

"The infinite we do right away, the finite takes some more time."

(By the way, we "stole" this motto from the U.S. Navy - during World War II one of their mottos was "The difficult we do right away, the impossible takes some more time".) To our motto (Ulam's and mine) I just want to add that the finite takes in fact very much more time - perhaps more than the lifetime of the universe.

In what follows I will give proofs only if the published proofs are hard to find, are excessively complicated, or are not quite accurate. In any case, I will give as complete references as possible. The first part of this survey will give an overview of several problems without going into too much detail. In the second part I will discuss one or two of the problems in some detail and also will use somewhat more set theory.

§1.

The first problem I want to consider is a clear-cut case of a problem in which the finite case is very much more difficult than the infinite one.

THEOREM 1. *Let* E_k *be* k-*dimensional Euclidean space,* S *a subset of* E_k *with* $|S| = m \geq \aleph_0$. *Then* S *has a subset* S_1 *with* $|S_2| = m$ *such that all the distances between points of* S_1 *are distinct.*

Although I first proved Theorem 1 about 30 years ago, I will give the proof here in some detail since the published proof is obscure and not accurate. The main point is that the continuum hypothesis is not assumed and in fact the theorem is almost trivial if m is a regular cardinal (i.e. if m is not the sum of fewer smaller cardinals). It will be clear from our proof where the simplifications occur if m is assumed to be regular. Let $cf(m) = n (n \leq m)$ be the smallest cardinal for which m is the union of n smaller cardinals. Assume that our theorem holds if $|S| \leq p < m$ and also assume that it holds if

$|S| = m$ and S is situated in a space of dimension less than k. By a subspace of E_k in this proof we will mean a hyperplane or hypersphere of E_k. Assume now that r is the smallest integer, $1 \leq r \leq k$ for which there are n r-dimensional subspaces P_α, $1 \leq \alpha < \omega_n$ such that $|\bigcup P_\alpha| = m$. Let $S_\alpha = P_\alpha \cap S$ and $|S_\alpha| = p_\alpha$. We can assume without loss of generality that p_α is an increasing function of α, that each p_α is regular, and that $p_\alpha \geq n$ for each α. All of these assumptions represent standard "tricks of the trade" when dealing with singular cardinals. Further, we can assume by our induction assumptions that all the distances in the set S_α are distinct, and this for every α.

First we prove that there is a subsequence P_{α_ν}, $1 \leq \nu < \omega_n$ of the P_α's such that no two elements of the subsequence are othogonal. It was pointed out to me by Bollobás and others that this step is missing in my original proof. Consider the ordinals $1 \leq \alpha < \omega_n$; join α_1 to α_2 if the corresponding subspaces P_{α_1} and P_{α_2} are orthogonal. Observe that there are at most k subspaces which are pairwise orthogonal. It immediately follows then from a theorem of Dushnik and Miller that there is a family of power n of subspaces no two of which are orthogonal. In the language of partition calculus the theorem we use can be expressed as $n \rightarrow (n,k)^2$, k finite. The theorem of Dushnik and Miller asserts that the theorem remains true if $k = \aleph_0$.

Thus we can assume that no two of our subspaces are orthogonal. Clearly we can also assume that the subspaces P_α are minimal in the sense that if P'_α is a proper subspace of P_α, then $|P'_\alpha \cap S| < p_\alpha$. To see this, observe that if this would not hold then we would simply replace P_α by P'_α and in a finite number of steps this replacement process would terminate.

We can now complete the proof as was done in my paper. In fact we shall prove that for every α, $1 \le \alpha < \omega_n$ there are sets $S'_\alpha \subset S_\alpha$ with $|S'_\alpha| = |S_\alpha| = p_\alpha$ such that all distances in $S' = \bigcup_{\alpha=1}^{\omega_n} S'_\alpha$ are distinct. The supposition that $|S'_\alpha| = p_\alpha$ would then give that $|S'| = m$ and our proof would be complete. What is left then is to construct our sets S'_α and we do this by transfinite induction. Suppose then that we have already constructed sets S'_α for each $\alpha < \beta < \omega_n$. We use these sets S'_α and points z_γ defined below to define $S'_\beta \subset S_\beta$. Suppose we have already found points z_γ, $1 \le \gamma < \delta < \omega_\beta$ which have the following properties. First of all, the distances in the set $(\bigcup_{\alpha<\beta} S_\alpha) \bigcup (\bigcup_{\gamma<\delta} z_\gamma)$ are all distinct. Further, none of the perpendicular bisectors of two points in this set contains a subspace P_α, and no subspace P_α is on a sphere whose center is one of our points. In other words, no P_α is equidistant from one of our points, and not all points of P_α can be equidistant from two of our points. Finally, if z is a point of our set and P_α one of our subspaces, then if $Q_\alpha(z)$ is the locus of our points y such that P_α is equidistant from z and y (i.e. the perpendicular bisector of z and y contains P_α) then none of our P's is contained in $Q_\alpha(z)$. These conditions, of course, exactly mean that we can find a $z_\delta \in S'_\beta$ such that $(\bigcup_{\alpha<\beta} S_\alpha) \cup (\bigcup_{1\leq\gamma\leq\delta} z_\gamma)$ have all of their distances distinct. To complete our transfinite induction then, we have to show that we can choose our z_δ so that our three conditions are satisfied. Since no two of our P'_α are orthogonal, we know that our z_δ is excluded from fewer than p_β subspaces none of which can contain P_β. Using the minimality of P_β we know that the intersection of P_β with this subspace meets S in a set of power less than p_β and by the regularity of p_β we see that z_δ can be chosen to satisfy all three of our conditions.

In Hilbert space the situation is completely different. Several of us, Oxtoby, Kakutani, L. M. Kelly, Nordhaus, and I observed that one can find a subset of Hilbert space of power c such that every distance is rational. Trivially, one can find a countable set in Hilbert space which determines only one distance, but every uncountable set determines infinitely many distances.

Now let us investigate the finite case. Let $f_k(n)$ denote the largest integer so that if $x_1,\dots,x_n$ are any n distinct points in E_k, then there are always $f_k(n)$ of them such that all the distances between these $f_k(n)$ points are distinct. The exact determination of $f_k(n)$ seems hopeless, and I cannot even get an asymptotic formula for $f_k(n)$. However, it is not hard to show that there are constants ε_k and ε_k' so that

$$(1) \qquad c_k n^{\varepsilon_k} < f_k(n) < c_k n^{\varepsilon_k'}$$

where ε_k and ε_k' tend to 0 as k tends to infinity. Perhaps

$$(2) \qquad \lim_{n\to\infty} \log f_k(n)/\log n = 1/(k+1).$$

But (2) is known only for k=1 and there is no real evidence for its truth.

Let us discuss the case where k=1. A plausible conjecture is

$$(3) \qquad f_1(n) = (1+o(1))n^{\frac{1}{2}}$$

The upper bound for $f_1(n)$ follows from the following result of Turán and myself: Let $1 \le a_1 < \ldots < a_k \le n$ be a sequence of integers so that the differences (i.e. distances) a_j-a_i are all distinct. Then

$$g_1(n) = \max k = (1+o(1))n^{\frac{1}{2}}$$

and in fact we conjecture that

$$(4) \qquad g_1(n) = n^{\frac{1}{2}}+O(1).$$

I offer 500 dollars for a proof or disproof of (4). It is reasonable to conjecture that $g_1(n) = f_1(n)$ or in other words if $\{x_1,\ldots,x_n\}$ is any set of real numbers, one can always find $g_1(n)$ of them so that all distances between them are distinct. No proof of this plausible conjecture is in sight. A very beautiful and general result of Komlós, Sulyok, and Szemerédi only gives $f_1(n) > cn^{\frac{1}{2}}$ for a certain $c > 0$. Thus, to summarize, we know that

$$(5) \qquad cn^{\frac{1}{2}} < f_1(n) < (1+o(1))n^{\frac{1}{2}}$$

and we conjecture that

$$(6) \qquad f_1(n) = g_1(n) = n^{\frac{1}{2}}+O(1).$$

I'll give 500 dollars for cleaning up (6) - I am, however, being deliberately vague. A counterexample for a particular value of n (to $f_1(n) = g_1(n)$) would not be so interesting, but I would certainly pay the 500 dollars if someone shows that $f_1(n) \neq g_1(n)$ holds for infinitely many n. And if at the same time he also proves that $f_1(n)/g_1(n) \to 1$ (I am really sure that this conjecture holds), I will probably pay an additional 500 dollars (that is, if I live long enough and have the money).

Very little is known about $f_k(n)$ for $k \geq 2$; one could guess that the extremal configuration is highly regular, but no proof of this is in sight. There is a paper of Richard Guy and myself, however, which contains some preliminary results.

Denote by n_k the smallest integer for which $f_k(n_k) = 3$. That $n_1 = 4$ is trivial. I've observed that

$n_2 = 7$, and Croft proved that $n_3 = 9$. It seems likely that $n_k < ck^2$, but as far as I know it is not even known whether $n_k^{1/k} \to 1$.

A related question has recently been nearly completely solved by Larman, Rogers, and Seidel. Let $S_k^{(r)}$ be a set in k-dimensional space which determines at most r distances. Trivially, max $|S_k^{(1)}| = k+1$ and they proved that

$$\max |S_k^{(2)}| = k^2/2 + O(k).$$

Their method, no doubt, will give that for fixed r

$$c_2 k^r < \max |S_k^{(r)}| < c_1 k^r.$$

The vertices of the k-dimensional cube determine k distinct distances; perhaps max $|S_k^{(r)}|$ is not much larger than 2^k. For further problems and results of this kind see my paper in Annali di Mat and my forthcoming book with Purdy.

I hope I've convinced the reader that problems on infinite sets can be much simpler than problems concerning finite sets. Often the reason is that for infinite cardinals, $m^2 = m$ holds.

Now we return to infinite problems. Kakutani and I proved that $c = \aleph_1$ is equivalent to the statement that the real line is the union of countably many Hamel bases. First I'll show that if $c = \aleph_1$ then the real line is the union of countably many Hamel bases. Let $\{a_\alpha\}$, $1 \le \alpha < \omega_1$ be a Hamel base and let S_α be the set of real numbers $\sum c_\beta a_\beta$ where the c_β are rational and $\max \beta = \alpha$. Since $c = \aleph_1$, $|S_\alpha| = \aleph_0$. Enumerate the elements of S_α in an ω sequence $\{x_n^\alpha\}$, $n = 1,2,\ldots$. Define $H_n = \{x_n^\alpha\}$ where α runs through the ordinals less than ω_1. Clearly, the H_n give our required decomposition of the reals into $\aleph_0$ Hamel bases. This proof (which is a bit simpler than the one given in our paper) is similar to our old (1938) unpublished result with Tuhey: The complete graph of power $\aleph_1$ is the countable union of trees.

Of course, our result with Kakutani implies that if $c = \aleph_1$ then the real line is the union of sets S_n, $n = 1,2,\ldots$ such that all the S_n have all their distances distinct (i.e. any four points of S_n determine six distinct distances). I conjectured that if $c = \aleph_1$ then E_k is the union of $\aleph_0$ sets S_n so that each of the S_n have all their distances distinct. This conjecture was proved by R. O. Davies for $k = 2$, but as far as I know, $k > 2$ is still open.* Ceder proved that E_k is the countable union of sets S_n none of which contains an equilateral triangle.

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*<u>*Added in proof:*</u> I just (May 1979) received a letter from K. Kunen and he has proved the conjecture for all k.

I will now move on to some curious geometrical and measure theoretical problems. It is not difficult to see that if S is a plane set of infinite planar measure, then for every positive real number a, S contains three points x_1, x_2, x_3 such that the area of the triangle $\{x_1, x_2, x_3\}$ is a. The proof is an easy consequence of the Lebesgue density theorem and is left as an exercise to the reader. In fact I published this as a problem in the <u>Matematikai Lapok</u> and one of the readers proved a slightly stronger theorem: It suffices to assume that at least one line intersects S in a set of positive linear measure and that there are points arbitrarily far from this line. In fact it is easy to see that our triangle of area a can be taken to be isosceles or right angled. More generally and slightly vaguely, besides specifying the area of the triangle, one additional condition can be imposed on the triangle and still obtain the result. On the other hand it is very easy to see that there is a set S in the plane of infinite planar measure which contains no equilateral triangle of unit area.

The following question seems interesting and perhaps difficult: Is it true that there is an absolute constant C so that if S has planar measure greater than C then S contains the vertices of a triangle area 1? If S is the set $|\zeta| < 2.3^{-3/4}$ then S does not contain a triangle of area 1 (we use the well known result in elementary geometry that the triangle of largest area inscribed in the circle is equilateral). The area of S is $4\pi\ 3^{-3/2}$ and perhaps this is the correct value of C. I have no real evidence for this conjecture. These problems can clearly be stated for higher dimensions as well.

Here is another curious problem on measurable sets. Let S be a set of positive measure on the line and A any finite subset of the line. Then it easily follows from the Lebesgue density theorem that S contains a set similar to A (i.e. contains a set A' which can be transformed into A by a fractional linear transformation). This result is substantially due to Steinhaus and has often been rediscovered. I have conjectured for a long time that if A is any infinite set on the line then there always is a subset S of the line of positive measure which does not contain a set similar to A. By the way, we can assume without loss of generality that A is a sequence of positive numbers tending to 0. If my conjecture is correct then one can further ask the following: Given a countable set A of [0,1] determine (or estimate) the largest possible measure of a subset S of [0,1] which does not contain a set similar to A.

Now I'll state a problem in geometric number theory. Denote by d(u,v) the distance from u to v. Let $N(x,\delta)$ be the maximum number of points $P_1,\ldots,P_n$ which can be chosen in the circle of radius x so that

$$\min_k |d(P_i,P_j)-k| \geq \delta,\ 1 \leq i < j \leq n \text{ and } k \text{ an integer}.$$

I conjectured that for every $0 < \delta < 1/2$ that

$$N(x,\delta) = o(x).$$

and on the other hand I conjectured that there is a $\delta_o > 0$ such that

$$\lim_{x\to\infty} N(x,\delta_o) = \infty.$$

The analogous problems are trivial in one dimension and perhaps interesting new complications arise if the dimension is greater than two. The first of these conjectures was proved by Sárközy who showed that

$$N(x,\delta) < \frac{4\cdot 10^4}{\delta^3} \frac{x}{\log\log x}.$$

The second conjecture was proved by Graham who showed that

$$N(x,1/10) > \frac{\log x}{10}.$$

Sárközy then improved this to

$$N(x,1/10) > x^c$$

where $c > 0$ is an absolute constant. Sárközy further proved that for every $\varepsilon > 0$ there is a $\delta(\varepsilon)$ such that if $\delta < \delta(\varepsilon)$ and $x > x_o(\varepsilon,\delta)$, then

$$N(x,\delta) > x^{1/2-\varepsilon}$$

The exact magnitude of $N(x,\delta)$ is not known and is perhaps difficult to determine.

Let $f(n)$ be the largest integer for which there are n distinct points $P_1,\ldots,P_n$ in the plane for which there are $f(n)$ pairs (P_i,P_j) satisfying the condition that $d(P_i,P_j) = 1$. It is known that

$$n^{1+c/\log\log n} < f(n) = o(n^{3/2}).$$

Once again I refer you to our forthcoming book with George Purdy where these and related questions are extensively discussed.

Before closing this section I'll state a few problems in a new subject which my collaborators and I call Euclidean Ramsey theory. A set S in a finite dimensional Euclidean space is called *Ramsey* if to every k there is an n_k such that if E_{n_k} is colored by k colors (or in other words, E_{n_k} is decomposed into k disjoint sets A_i, $1 \le i \le k$), then S can be monochromatically imbedded into one of the A_i's. We proved that every brick (i.e. every set of vertices of a rectangular parallelopiped) is Ramsey and on the other hand we showed that every set which is Ramsey can be inscribed in a sphere.

The most interesting and challenging problems are: Are the obtuse angled triangles Ramsey? Is the regular pentagon Ramsey?

Let $S_1 \cup S_2 = E_2$. Is it true that if T is any triangle (with the possible exception of equilateral triangles of one fixed height) then either S_1 or S_2 contains the vertices of a triangle congruent to T? Many special cases of this startling conjecture have been proved by us and Schader, but so far the general case eludes us.

Let S be a set of points in the plane such that no two points of S are at a distance of one. We conjectured that the complement of S contains the vertices of a unit square. This conjecture was proved by R. Juhász. She in fact showed that if $\{x_1, x_2, x_3, x_4\}$ is any set of four points, the complement of S contains a congruent copy of $\{x_1, x_2, x_3, x_4\}$. It is not clear at present if this remains true for five points; indeed she showed that there is a k so that the result fails for k points.

Clearly very many more problems can be stated and I hope more people will work on this subject in the future and our results will soon become obsolete.

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§2.

First I'll give a proof of the second part of my theorem with Kakutani (see page 120). In fact, the theorem I'll prove is slightly stronger.

THEOREM 2. *Suppose* $c > \aleph_1$ *and* $E_1 = \bigcup_{n=1}^{\infty} S_n$. *Then there is at least one* n *such that the distances determined by* S_n *are not all different*.

What we shall prove is that there are four points in S_n which determine at most four different distances, and to do this we use the following lemma due to Hajnal and myself.

Suppose $|A| = \aleph_2$, $|B| = \aleph_1$, and $A \cap B = \emptyset$. Let $K(A,B)$ denote the complete bipartite graph spanned by $A \cup B$ (i.e. the vertices of the graph $K(A,B)$ are the elements of $A \cup B$ and every $x \in A$ is joined by an edge to every $y \in B$). Now, if the edges of $K(A,B)$ are colored with $\aleph_0$ colors, there is a monochromatic C_4 (i.e. a circuit of length four all of whose edges have the same color).

I'll prove this lemma of Hajnal and myself in full detail since I cannot give an exact reference to it. Denote the edges of the i-th color by G_i, $i = 1,2,\ldots$ and observe that every vertex $x \in A$ has valency (or degree) $\aleph_1$ in at least one of the graphs G_i. Since $|A| = \aleph_2$ there clearly is an i such that there are $\aleph_2$ vertices $x \in A$ which have valency $\aleph_1$ in G_i. For this i, and for each such $x \in A$ there are $\aleph_1$ vertices in B which are joined to x and we denote this set of vertices by $S(x)$. Note that $S(x) \subset B$ and $|S(x)| = \aleph_1$. Now consider all the pairs of $S(x)$ for all of the $\aleph_2$ vertices $x \in A$ mentioned above. There are only $\aleph_1$ pairs of B and thus the same pair must be joined to $\aleph_2$ elements of A which gives us our C_4 and indeed gives a monochromatic $K(\aleph_2, 2)$. In fact, Hajnal and I proved the following result: Let $m > \aleph_0$ (i.e. m is not the union of $\aleph_0$ smaller cardinals). Decompose $K(m, \aleph_1)$ as the union of countably many graphs G_i, $i = 1, 2, \ldots$. Then for at least one i and for every $\alpha < \omega$, G_i contains a $K(m, \alpha)$. The proof is very similar to the one given here and can be left to the reader (the reader must of course be familiar with the standard arguments in combinatorial set theory, also known as infinitary combinatorics).

From our result with Hajnal it now follows immediately that if $c > \aleph_1$ and $E_1 = \bigcup_{n=1}^{\infty} S_n$ then for at least one n, S_n contains four points which determine exactly four different distances. To see this, let H be a Hamel basis (The fact that $|H| > \aleph_1$ follows immediately from the assumption that $c > \aleph_1$), $A \subset H$, $B \subset H$, $A \cap B = \emptyset$, $|A| = \aleph_2$, and $|B| = \aleph_1$. Consider the set Z of distinct real numbers $x+y$, $x \in A$ and $y \in B$. This set can be represented by the edges of the bipartite graph $K(A,B)$. The sets $Z \cap S_n$ define the graphs G_n and give a decomposition of $K(A,B)$ into countably many graphs. By the lemma, there is an n such that G_n contains a rectangle - in other words, there are four real numbers $x_1 \in A$, $x_2 \in A$, $y_1 \in B$, $y_2 \in B$ so that all four of the numbers

(1) $$x_1+y_1, \; x_1+y_2, \; x_2+y_1, \; x_2+y_2$$

are in S_n and determine at most four distances - as stated.

Next we show that if $c = \aleph_2$ there is a decomposition of E_1 into $\aleph_0$ sets S_n, $n = 1,2,\ldots$ such that the distances between the points of S_n are all distinct except for relations of the form (1). I realize that this informal formulation is not as clear as it perhaps should be, but it will be clear to the reader from the construction of our sets S_n.

Again, let $H = \{a_\alpha\}$, $1 \leq \alpha < \omega_2$ be a Hamel basis. Thus if y is a real number there is a finite set of rational numbers $\{c_\alpha\}$ such that the unique representation of y with respect to H is

$$y = \sum_{\alpha} c_\alpha a_\alpha . \tag{2}$$

Let $h(y) = \beta$ be the largest α so that a_α appears in the representation of y. Then $h(y) < \omega_2$. Denote by R_β the set of real numbers y with $h(y) = \beta$. Clearly, $|R_\beta| \leq \aleph_1$ and thus R_β can be decomposed into countably many sets $S_n^{(\beta)}$ such that all the distances between points of $S_n^{(\beta)}$ are distinct (e.g. use the method of Kakutani and myself described earlier). Finally put

$$S_n = \bigcup_{\beta} S_n^{(\beta)} .$$

The reader can easily convince himself that this decomposition satisfies our requirements. This method was used several times by Hajnal and myself (but very well might have been used earlier).

A special case of an unpublished theorem of Elekes, Hajnal, and myself states that if $|A_i| = \aleph_i$, $1 \le i \le r$ are r disjoint sets and if we split the r-tuples $(x_1,\ldots,x_r)$, $x_i \in A_i$ into $\aleph_0$ classes, then there are 2r elements $x_i^{(1)}$, $x_i^{(2)} \in A_i$, $i = 1,\ldots,r$ such that all the 2^r r-tuples $(y_1,\ldots,y_r)$, $y_i = x_i^{(1)}$ or $x_i^{(2)}$ are in the same class. This is, of course, a generalization of our theorem with Hajnal for $r = 2$. Our theorem implies, just as in the case $r = 2$, that if $c \ge \aleph_r$ and we decompose E_1 into countably many sets S_n, $n = 1,2,\ldots$, then for at least one n there are 2r real numbers $x_i^{(1)}$, $x_i^{(2)}$, $i = 1,\ldots,r$ such that each of the 2^r sums

$$(3) \qquad \sum_{i=1}^{r} y_i, \text{ where } y_i = x_i^{(1)} \text{ or } x_i^{(2)}$$

is in S_n. Thus, one of the S_n contains 2^r points which determine $(3^r-1)/2$ distances. Using the method of Hajnal and myself it is not hard to see that if $c = \aleph_1$ then E_1 can be written as the union of $\aleph_0$ sets S_n, $n = 1,2,\ldots$ such that for every n, the set of distances for S_n satisfy (3) and no other relations. In light of this, perhaps the study of the following question is of some interest: Denote by $f(k,r,t)$ the smallest integer ℓ such that if we assume that $c = \aleph_r$ and decompose E_k into countably many sets S_n, $n = 1,2,\ldots$

then there always are t points $x_1, \ldots, x_t$ in one of the S_n such that $\{x_1, \ldots, x_t\}$ determines at most ℓ distinct distances. Using (3), $f(1,r,t)$ can easily be determined, though the explicit formula for $f(1,r,t)$ seems complicated. At present, nothing can be done in the case that $k > 1$. This is true even if we assume $c = \aleph_1$ $(r=1)$ since we do not even know if E_3 can be decomposed into $\aleph_0$ sets none of which contains an isosceles triangle. In other words we don't know if $f(3,1,3)$ is 2 or 3, though by a result of Ceder we know it is greater than 1.* Perhaps using the method of Davies one can determine $f(2,r,t)$ for every r and t. Throughout this discussion we have assumed that $r < \omega$ (i.e. $c < \aleph_\omega$). It seems certain that if we drop this assumption so that $c \geq \aleph_\omega$, then

$$(4) \qquad f(k,c,t) = \min_{r \to \infty} f(k,r,t)$$

but (4) remains unproved.

Interesting and probably difficult finite problems remain. Let me state a few. Suppose there are n points in the plane: What is the maximum number of equilateral or isosceles triples? What is the maximum number of quadruples which determine exactly five distinct distances? Purdy and I have some preliminary results on

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**Added in proof:* The new result of Kunen yields that $f(k,1,t) = \binom{t}{2}$.

these questions and hope to write some more about them in our forthcoming book.

Now assume that $c > \aleph_1$ and let S_n be a set of real numbers such that all sums $x+y$, $x \in S_n$, $y \in S_n$ are distinct (or in other words the distances between points of S_n are distinct). Then it is not difficult to show that $\overline{\bigcup_{n=1}^{\infty} S_n}$ (i.e. the complement of the union of countably many S_n) contains a translate of an $\aleph_1$ dimensional linear subspace of the reals. That is, there is a set of $\aleph_1$ rationally independent numbers $\{b_\alpha\}$, $1 \leq \alpha < \omega_1$ such that for some t all numbers of the form

$$t + \sum_{\beta} r_\beta b_\beta, \ r_\beta \text{ is rational and the sum is finite,}$$

are all contained in $\overline{\bigcup_{n=1}^{\infty} S_n}$.

To prove this, first observe that from our proof with Hajnal given at the beginning of this section we easily obtain that there is a set A with $|A| = \aleph_2$ such that for every rational r and $a_\alpha \in A$, ra_α is not in $\overline{\bigcup_{n=1}^{\infty} S_n}$. Let B be any set of rationally independent reals whose power is $\aleph_1$. A real number $a_\alpha \in A$ is called bad if there are $\aleph_1$ reals of the form

$$r_\alpha a_\alpha + \sum_{\beta} c_\beta b_\beta, \ r_\alpha \text{ and } c_\beta \text{ rational and } b_\beta \in B$$

in $\overline{\bigcup_{n=1}^{\infty} S_n}$. First note that there are at most $\aleph_1$ bad a_α's. To see this notice that there are at most countably many choices for r_α, n, and the c_β's and thus we can assume that there are $\aleph_2$ a_α's for which they are the same. For each of these $\aleph_2$ a_α's choose two numbers

$$r_\alpha a_\alpha + \sum_\beta c_\beta b_\beta \text{ and } r_\alpha a_\alpha + \sum_\beta c_\beta b_{\beta'}, \ (r \neq 0)$$

which are both in the same S_n. Now finally, there are only $\aleph_1$ choices for $\{b_\beta\}$ and $\{b_{\beta'}\}$, and thus since the number of the a_α's was $\aleph_2$ there are two of them (in fact $\aleph_2$ of them) a_α and $a_{\alpha'}$ which get the same set $\{b_\beta\}$ and $\{b_{\beta'}\}$. Hence, the four numbers

$$r_\alpha a_\alpha + \sum_\beta c_\beta b_\beta, \ r_\alpha a_\alpha + \sum_\beta c_\beta b_{\beta'}, \ r_\alpha a_{\alpha'} + \sum c_\beta b_\beta, \ r_\alpha a_{\alpha'} + \sum c_\beta b_{\beta'},$$

all belong to the same S_n. But this is clearly impossible since the sum of the first and the fourth equals the sum of the second and the third. Therefore there are only $\aleph_1$ bad a_α's and so $\aleph_2$ of the a_α's are not bad. But if a_α is not bad, then there are at most $\aleph_0$ b_β's for which there are rational numbers $\{r_\alpha, c_\beta\}$ such that $r_\alpha a_\alpha + \sum_\beta c_\beta b_\beta$ is in $\overline{\bigcup_{n=1}^\infty S_n}$. Omit these b_β's. Thus, finally, we have a set of b_β's of power $\aleph_1$ and an a_α such that for every non-zero rational r_α and arbitrarily rational $\{c_\beta\}$,

$$r_\alpha a_\alpha + \sum_\beta c_\beta b_\beta$$

is not in $\overline{\bigcup_{n=1}^\infty S_n}$ (observe that if every c_β is zero, then

$r_\alpha a_\alpha$ is not in $\overline{\bigcup_{n=1}^{\infty} S_n}$). Thus our assertion is proved.

Perhaps this result can be strengthened in two ways. First of all, $\overline{\bigcup_{n=1}^{\infty} S_n}$ perhaps contains all numbers of the form $\sum_\beta r_\beta b_\beta$ where the r_β are rational and $\{b_\beta\}$ are chosen from a set of $\aleph_1$ rationally independent numbers. (In other words, the additive constant t may be superfluous.) Secondly, perhaps $\aleph_1$ can be replaced by $\aleph_2$. I do not think the latter is likely, but have not yet found a counterexample.

A few years ago I asked: Let S be a set of real numbers for which all sums $x+y$, $x \varepsilon S$, $y \varepsilon S$ are distinct. Is it true that $\overline{S}$ contains an infinite arithmetic progression? Baumgartner proved this, and my proof given above borrows from Baumgartner's unpublished proof.

Hilbert space behaves in a completely different way than the Euclidean spaces E_k. Hajnal and I easily showed that one can give c points in Hilbert space such that all triangles are isosceles and acute angled. Also, there are c points in Hilbert space such that all the distances are rational.

I asked two further questions: Is there a set S of power c in Hilbert space such that every subset S_1 of S with $|S_1| = c$ contains an equilateral triangle? Also, is there such a set S such that every subset S_1 of S with $|S_1| = c$ contains an infinite dimensional regular simplex?

L. Pósa answered both questions affirmatively; for the first he used no hypothesis concerning c; for the second he had to assume that $c = \aleph_1$.

Kunen and I proved that if $c > \aleph_1$ then the union of $\aleph_0$ rationally independent sets always has inner measure 0. As the proof of this has never been published, I shall outline the proof here. First note that every set of positive inner measure contains a perfect set of positive measure. We then prove that if T is a perfect set of positive measure, then there are perfect sets P and Q so that $P+Q \subset T$. We suppress the details of this. Our result with Hajnal on the decomposition of K(A,B) with $|A| = \aleph_2$ and $|B| = \aleph_1$ then completes our proof.

§3. Some final remarks.

Fajtlowitz and I observed that if $c = \aleph_1$ then the plane can be decomposed into $\aleph_0$ sets $\bigcup_{n=1}^{\infty} S_n$ such that no three points of any S_n determine a right angle. On the other hand, if $c > \aleph_1$ then at least one of the S_n must contain a rectangle.

Assuming $c = \aleph_1$, Sierpinski decomposed the real line into two sets, A_1 and A_2 such that any translation of A_1 intersects A_2 in at most a countable set.
P. Lax and I showed that this is best possible -

namely, if $E_1 = A_1 \cup A_2$, $|A_i| = c$, and $m < c$, then there is a real number t so that $|A_1+tA_2| \geq m$.

I conjectured, and Peter Komjáth proved that if $c = \aleph_1$ then there are sets A and B such that $E_1 = A \cup B$ and for every real number z the number of solutions of $x+y = z$, x and $y \in A$ or x and $y \in B$ is countable.

R. L. Graham recently proved that if we decompose E_2 into finitely many sets S_i, $E_2 = \bigcup_{i=1}^{n} S_i$, then for at least one i, the set S_i contains the vertices of triangles of any given area. Graham's proof will soon appear in the Journal of Combinatorial Theory. Graham and I tried to extend this result to countable decompositions of E_2. There appear to be two possibilities: 1. There is a constant c, which perhaps depends on the decomposition such that for at least one i and for every $\alpha < c$ there is a triangle $\{x,y,z\}$ in S_i of area α. (A weaker form of this conjecture would be that to every $\alpha < c$ there is an i such that there is a triangle $\{x,y,z\}$ of area α whose vertices are all in S_i). 2. Assume that every S_i is unbounded. Then there is an i such that S_i contains triangles of all areas. It is quite possible, of course that none of these conjectures holds.

In a forthcoming triple paper, Kunen, Mauldin and I prove, among others, the following theorem: If $c = \aleph_1$, there there is a set, A, of real numbers such that $|A| = c$ and for every set B of measure zero, A+B also has measure 0 (A+B is the set of all a+b where $a \in A$ and $b \in B$). Now, Kakutani and Oxtoby have obtained far reaching extensions of Lebesgue measure (countable additivity and congruence invariance are perserved). Does our result remain true for these extensions? This question should, perhaps, be interpreted to mean that there is a further extension in which all the sets A+B are of measure 0. The last two conjectures are rather new and we have had no time to think them over, so I must ask for the indulgence of the reader if they turn out to be either trivial or false.

S. Kakutani and Oxtoby, Construction of non-separable invariant extension of the Lebesgue measure space, Annals of Math. (2), 52 (1950), 580-590.

P. Erdös, Some remarks on set theory, Annals of Math. 44, (1943), 693-696.

P. Erdös, A. Hajnal and R. Rado, Partition relations for cardinal numbers, Acta Math. Hung. Acad. Sci. 16, (1965), 93-196. (This paper contains many results and problems and arguments related to the one used by Hajnal and myself in this survey.)

H. Halberstam and K. F. Roth, Sequences, Oxford University Press 1966. (The result of Turán and myself (on a_i+a_j) and many interesting questions in additive and combinatorial number theory are discussed here.)

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