Roy O. Davies, Department of Mathematics, The University,
Leicester, LE1 7RH, England

Symmetric Sets are Measurable

Evans and Weil [1] call a set $E \subseteq \mathbb{R}$ symmetric if its characteristic function f satisfies the condition: for every $x \in \mathbb{R}$ there exists δ_x such that $f(x+h) = f(x-h)$ for $0 < h < \delta_x$; and they ask whether if E is symmetric it must be measurable. The answer is Yes, and indeed either E or its complement must contain a co-countable open set.

Hausdorff [4] called a real function f symmetrically continuous if for every $x \in \mathbb{R}$, $f(x+h) - f(x-h) \to 0$ as $h \to 0$, and he asked whether a symmetrically continuous function can have uncountably many discontinuity points. In this connection Fried [3] proved that it has a dense set of continuity points. Now the characteristic function f of a symmetric set is evidently a symmetrically continuous function, so by Fried's theorem it has a continuity point. For a characteristic function, this means that there is a non-degenerate open interval on which f is constant. Let c be an interior point of this interval, and let d be the supremum of points $x \in \mathbb{R}$ such that f is constant on a co-countable open subset of (c, x). We see that $d = +\infty$ because otherwise the symmetry property applied at d leads to a contradiction. We argue similarly on the left of c, and conclude that f is constant on a co-countable open subset of $\mathbb{R}$.

Readers may be interested in what Fried's argument reduces to in the case of our characteristic function f. Let B_n

denote the set of x such that $1/n$ will serve for δ_x. By Baire's theorem, there exists a non-degenerate open interval I such that B_n is dense in I for some n, and we may suppose that the length of I is less than $1/n$. Without loss of generality we may also suppose that the midpoint of I is 0 and $0 \in B_n$. Let $2x$ be an arbitrary point of I, and choose $x' \in B_n$ so close to $\frac{1}{2}x$ that $|2x' - x| < \delta_x$ and $2x' \in I$, $2x - 2x' \in I$. Now by successive applications of the symmetry condition about the respective points $x', x, 0, x'$, we see that

$$f(0) = f(2x') = f(2x - 2x') = f(2x' - 2x) = f(2x) .$$

Thus f is constant in I.

By the way, Hausdorff's problem was solved finally by Preiss [5], who constructed a symmetrically continuous function possessing uncountably many discontinuity points. Preiss also solved another natural problem by proving that if f is symmetrically continuous then it is continuous almost everywhere (and therefore measurable, in particular). He did this via showing that $\phi = \min(\operatorname{osc} f, 1)$ is also symmetrically continuous and applying Fried's theorem. The paper of Foran [2] should be read in the light of [5].

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