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Locally Symmetric Functions

Throughout we let f denote an arbitrary function from the real line R into an arbitrary set A. We call f locally symmetric if to each x in R there corresponds a $\delta_x > 0$ with

(1) $$f(x-h) = f(x+h) \quad \text{for} \quad 0 \leq h < \delta_x \ .$$

M. Foran [2] proved that a measurable locally symmetric function is constant except on a nowhere dense countable set. Here we prove the following stronger result.

Theorem. *If f is locally symmetric, then for some $q \in A$ the closure of the set $\{x: f(x) \neq q\}$ is countable.*

The proof of this theorem will be preceded by the proof of a lemma concerning the function d, where $d(x)$ is defined to be the supremum of the numbers δ_x that satisfy (1).

Lemma. *d is upper semicontinuous if f is locally symmetric.*

Proof. It suffices to prove that d is upper semicontinuous at 0. Suppose to the contrary that, for some constant $c > 0$, there are numbers x arbitrarily close to 0 with

(2) $$d(x) > d(0) + c \ .$$

Set $c_1 = (1/10)\min(c, d(0))$, and choose two numbers x_1 and x_2 satisfying (2) such that $x_1 < x_2$ and

(3) $$|x_j| \leq c_1 \quad \text{for} \quad j = 1 \text{ and } 2 \ .$$

Set $c_2 = x_2 - x_1$ $(c_2 \leq 2c_1)$ and consider any number h with

(4) $$d(0) \leq h < d(0) + c_2 \ .$$

To complete the proof, we shall show that $f(-h) = f(h)$ in contradiction with the definition of $d(0)$.

We introduce the notation $[p]_j$ to denote the reflection of any point p about the line $x = x_j$ $(j=1,2)$. Then in view of the definition of c_1 and the inequality $c_2 \le 2c_1$, it follows easily from (3) and (4) that

$$|x_1 - [h]_2| < |x_2 - [h]_2| = |x_2 - h| < d(0) + c \quad .$$

Consequently, since x_1 and x_2 satisfy (2), we have

$$|x_2 - h| < d(x_2) \quad \text{and} \quad |x_1 - [h]_2| < d(x_1) \quad ,$$

and it follows from the definition of d that

(5) $$f(h) = f([h]_2) = f([[h]_2]_1) \quad .$$

Similarly we deduce that

(6) $$f(-h) = f([-h]_1) = f([[-h]_1]_2) \quad .$$

Now direct computations yield

(7) $$[[h]_2]_1 = h - 2c_2 = -[[-h]_1]_2 \quad ;$$

furthermore, the first of these equalities, together with (4) and the inequality $2c_2 < d(0)$, gives

(8) $$0 < [[h]_2]_1 < d(0) \quad .$$

Then (5) - (8) and the definition of $d(0)$ imply that the desired equality $f(-h) = f(h)$ holds. □

Remark. The key to the preceding proof is the fact that the composition of two parallel reflections is a translation.

Proof of the theorem. Due to the Lemma and to the local symmetry of f, the sets

$$X_n = \{x: d(x) \ge 1/n\}$$

are closed and their union is R. By Baire's theorem, there is an n for which X_n contains an interval I. If

$$a < b < a + 1/n, \quad a, b \in I \ ,$$

then by applying the symmetry condition for $(a+b)/2$ we have $f(a) = f(b)$. This means that f is constant on I. Denote this constant by q, and set $Q = f^{-1}(q)$. Then the set $\dot{X}$ of condensation points of the set $X = R \setminus \mathrm{int}(Q)$ is either perfect or empty; suppose the former and let $I = (a,b)$ be a complementary interval to $\dot{X}$. (Note that $\dot{X} \neq R$, as $\mathrm{int}(Q) \neq \emptyset$.) At least one of a and b, say a, must be finite. From the symmetry condition for a we conclude that $(a-h,a) \cap \dot{X} = \emptyset$ for sufficiently small h; that is, a is an isolated point of $\dot{X}$, a contradiction. Therefore $\dot{X} = \emptyset$ and X is a countable closed set. □

Call a set locally symmetric if its characteristic function is locally symmetric. Then the following corollary provides an affirmative answer to Query 37 in [1].

Corollary. If S is a locally symmetric subset of R, then either $\overline{S}$ or $\overline{R \setminus S}$ is countable.

In closing, we note that the Theorem is best possible in the following sense: every set of countable closure is contained in a locally symmetric set of countable closure.

REFERENCES

[1] M.J. Evans and C.E. Weil, Queries, Real Analysis Exchange 3(1977), 107.

[2] M. Foran, Symmetric functions, ibid. 1(1976), 38-40.

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