Real Analysis Exchange Vol. 4 (1978-79)

Jimmie Lee Johnson, Department of Mathematics
University of Wisconsin-Milwaukee, Milwaukee, WI 53201

The Uniform Continuity of Certain Translation Semigroups

Let $L^2(R^+;K)$ be the Lebesgue space of square summable functions f on the positive reals with values in a separable Hilbert space K. That is, f satisfies

i. $\langle f(x),k\rangle$ is measurable, a.e.(x), for each $k \in K$,

ii. $\int_0^\infty ||f(x)||^2 dx < \infty$, where $||f(x)||$ denotes the norm of f(x) in K.

The inner product is given by $\int_0^\infty \langle f(x),g(x)\rangle dx$, where $\langle f(x),g(x)\rangle$ is the inner product in K. For each $h \geq 0$, we define the translation operator S_h by

$$S_h f(x) = f(x + h).$$

$\{S_h\}$ is a strongly continuous semigroup of operators, i. e. for each f, $||S_h f - f||$ converges to 0 as $h \to 0^+$. However, it fails to be uniformly continuous; that is, $||S_h - I||$ does <u>not</u> converge to zero as $h \to 0^+$. For example,

$$f_h(x) = \begin{cases} 1/\sqrt{h} & \text{for } 0 \leq x \leq h \neq 0 \\ 0 & \text{elsewhere.} \end{cases}$$

satisfies $||f_h|| = 1$, $S_h f_h = 0$, so that $||S_h f_h - f_h|| = 1$. Hence $||S_h - I|| \geq 1$ for all $h > 0$. But if S_h is restricted to multiples of $e_1(x) = e^{-x}$, then since $S_h e_1 = e^{-h} e_1$, we have $||S_h - I|| = |e^{-h} - 1| \to 0$ as $h \to 0^+$. Therefore, one may ask the following question: If L is a closed linear subspace of $L^2(R^+;K)$ satisfying $S_h(L) \subset L$

for all $h \geq 0$, then is S_h^L, the restriction of S_h to L, a uniformly continuous semigroup?

Define the infinitesimal generator of S_h^L as follows: $(D_L f)(x) = \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}$ for those $f \in L$ for which at which the limit exists a. e. The set of such f is dense in L. In fact, S_h^L will be uniformly continuous iff D_L is a bounded operator defined on all of L.

Let $H^2(U;K)$ be the Hardy space of functions analytic in the unit disk U with values in K satisfying:

$$\sup_{0 \leq r < 1} \int_0^{2\pi} ||f(re^{i\theta})||^2 d\theta < \infty .$$

By an extension of Fatou's Theorem to K-valued functions, the boundary value functions $f(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta})$ exist, and the inner product can be given by:

$$\langle f,g \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle f(e^{i\theta}), g(e^{i\theta}) \rangle \, d\theta .$$

Choose a basis $g_n k_m$, $n = 0,1,\ldots.;\ m = 1,\ldots..$ where $\{k_m\}$ is a fixed orthonormal basis for K and g_n is the Laguerre function of order n given by

$$g_n(x) = e^{x/2} \frac{d^n}{dx^n}(xe^{-x}).$$

Then $J: L^2(R^+;K) \to H^2(U;K)$ is defined by $J(g_n k_m)(z) = z^n k_m$ on the basis and is extended continuously to a unitary operator on L^2.

The backward shift T on $H^2(U;K)$ is defined by:

$$Tf(z) = \frac{f(z) - f(0)}{z} \quad \text{for } |z| < 1.$$

LEMMA: As unbounded operators on J(L),

$$JD_L J^{-1} = \frac{1}{2}(T + I)(T - I)^{-1}.$$

Hence, D_L is bounded as an operator on L iff T - I has an inverse on J(L). J. W. Moeller has determined the spectrum of restrictions of T to invariant subspaces in the case of scalar-valued functions. [4]

LEMMA: J(L) is invariant under T.

This can be done by factoring J through another Hardy space $H^2(P;K)$ where P is the open upper half plane. A function $f:P \to K$ is in $H^2(P;K)$ iff f is analytic in P and

$$\sup_{t>0} \int_{-\infty}^{\infty} ||f(s + it)||^2 ds < \infty .$$

Again the boundary value functions $f(s) = \lim_{t \to 0+} f(s + it)$ exist, a.e., and the inner product is given by:

$$<f,g> = \int_{-\infty}^{\infty} <f(s),g(s)> ds.$$

The Fourier integral transform $F(f)(u) = \int_0^{\infty} e^{iux} f(x)dx$ can be defined on $L^2(R^+;K)$ and maps L^2 isometrically onto $H^2(P;K)$.

Now W: $H^2(P;K) \to H^2(U;K)$ can be defined using a conformal mapping of P onto U in this manner:

$$Wg(z) = \frac{\sqrt{2\pi}}{(z - 1)} g\left(\frac{z + 1}{2i(z - 1)}\right) , \quad |z| < 1.$$

Then J = WF. Now let N denote the orthogonal complement of F(L) in $H^2(P;K)$. Then N is invariant under multiplication by exponentials of the form e^{ihu} for $h \geq 0$. Using approximation by trigonometric polynomials of such exponentials, N can be shown to be invariant under multiplication by $(2u + 1)(2u - 1)^{-1}$, so that W(N) is invariant under $T^{*}f(z) = zf(z)$, the adjoint of T. But $W(N) = WF(L)^{\perp} = J(L)^{\perp}$, so J(L) is invariant under T.

<u>REPRESENTATION THEOREM</u>: If M is a closed linear subpace of $H^2(U;K)$ which is invariant under T, then there exists an analytic function G_M defined on U satisfying

$$H^2(U;K) = M \oplus G_M H^2(U;K)$$

where: i. if dim K = 1, then G_M is a complex-valued <u>inner</u> function, i. e., $|G_M(z)| \leq 1$ for $z \in U$ and $|G_M(e^{i\theta})|$ is 1 a. e. G_M is unique up to multiplication by a number of unit modulus. [1]

ii. if dim K = n, then there exists H, a Hilbert space of dimension $\leq$ dim K such that $G_M(z)$: H → K is an operator with $||G_M(z)|| \leq 1$ and $G_M(e^{i\theta})$ is an isometry a. e. G_M is unique up to multiplication on the right by a unitary matrix. G_M is called <u>inner</u> if dim H = dim K so that $G_M(e^{i\theta})$ is unitary a. e. [3]

iii. if dim K = ∞, then $G_M(z)$:K → K is an operator with $||G_M(z)|| \leq 1$ and $G_M(e^{i\theta})$ is a partial isometry a. e. with a common initial space. That is, $G_M(e^{i\theta})$ is an isometry on the initial space and zero on its orthogonal complement. G_M is unique up to multiplication on the right by a partial isometry corresponding to a different choice of initial space. G_M is called <u>inner</u> if $G_M(e^{i\theta})$ is unitary a. e. [2]

SPECTRAL THEOREM: Let G_L be the function given by the representation theorem for the subspace J(L) and R(T,L) will denote the resolvent set for T restricted to J(L).

i. if $z \varepsilon U$, then $z \varepsilon R(T,L)$ iff $G_L(z^*)$ is invertible. That is, it is nonzero if dim K = 1 and is invertible as an operator on the appropriate Hilbert spaces otherwise.

ii. if $|u| = 1$, then $u \varepsilon R(T,L)$ iff $G_L(z)$ can be analytically continued across an arc of the unit circle containing u^*.

MAIN THEOREM: The restricted semigroup S_h^L is uniformly continuous iff the function G_L is an inner function and can be analytically continued across an arc of the unit circle containing z = 1.

Comment: Using $H^2(P;K) = F(L) \ominus Q_L H^2(P;K)$ where Q_L defined on P has similar properties to G_L, the analytic continuation at z = 1 can be replaced by the condition of being analytic at infinity.

REFERENCES

1. Beurling, A., On two problems concerning linear transformations in Hilbert space, Acta Math. 81 (1949), 239-255.

2. Halmos, P.R., Shifts on Hilbert space, Crelle's Journal 208 (1961), 102-112.

3. Lax, P.D., Translation invariant spaces, Acta Math. 101 (1959), 163-178.

4. Moeller, J.W., On the spectra of some translation invariant subspaces, J. Math. Anal. Appl. 4 (1962), 276-296.

Received September 7, 1978