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Λ-Bounded Variation: Recent Results and Unsolved Problems

Let $\Lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers such that $\Sigma\, 1/\lambda_n = \infty$. A real function f on an interval I is said to be of Λ-bounded variation (ΛBV) if $\Sigma\, |f(a_n)-f(b_n)|/\lambda_n < \infty$ for every sequence of non-overlapping intervals (a_n,b_n) in I. The supremum of such sums for $f \in \Lambda BV$ is necessarily finite and is called the total Λ-variation of f on I ($V_\Lambda(f)=V_\Lambda(f,I)$). The functions of harmonic bounded variation (HBV) are those in ΛBV with $\Lambda = \{n\}$. The literature on these classes is surveyed in our recent paper [1].

An interesting result which we inadvertently omitted from that survey is due to S. Perlman [2]. He has shown that the union of all ΛBV classes is the class of functions whose discontinuities are simple, and the intersection of all ΛBV classes is the class of functions of bounded variation (here he assumes $\lambda_n \nearrow \infty$). He shows also that these results cannot be improved by taking countable unions or intersections.

With Perlman, we recently investigated the dependence of the Λ-variation on the values of the function at points of discontinuity [3]. A function with a simple discontinuity at p is said to have an <u>internal</u>

<u>saltus</u> at p if

$$\liminf_{x \to p} f(x) \le f(p) \le \limsup_{x \to p} f(x) .$$

Our conclusions were:

1. If $f \in \Lambda BV$ and has an internal saltus at each point of discontinuity, then $V_\Lambda(f)$ is independent of the values of f at points of discontinuity.
2. If $f \in \Lambda BV$ and $g = f$ at the points of continuity of f, but has an internal saltus at each of its points of discontinuity, then $g \in \Lambda BV$ and $V_f \ge V_g$.

Suppose that $\Gamma = \{\gamma_n\}$ is a non-decreasing sequence of positive numbers such that $\Sigma\, 1/\gamma_n$ diverges. We have also shown that $\Lambda BV \subseteq \Gamma BV$ if and only if

$$\sum_1^n 1/\gamma_k = O\left(\sum_1^n 1/\lambda_k\right) .$$

In our survey paper [1] we showed that the Fourier coefficients of a function of ΛBV are $O(\lambda_n/n)$. This will appear in a note [4] where we also give a short proof that a theorem analogous to that of Dirichlet and Jordan holds for HBV functions and prove that the partial sums of the Fourier series of HBV functions are uniformly bounded. The principal result of that paper concerns complementary classes. Two classes of functions, K_1 and K_2, are said to be <u>complementary</u> if $f \in K_1$, $g \in K_2$

implies

$$\frac{1}{\pi}\int fg\,dx = \frac{1}{2}a_0a_0' + \Sigma(a_ka_k' + b_kb_k') ,$$

a_k, b_k being the Fourier coefficients of f and a_k', b_k' those of g. It is well known that L and BV are complementary. We have shown that

1. L and HBV are complementary
2. If ΛBV is not contained in HBV, then L and ΛBV are <u>not</u> complementary.

The first of these is quite straightforward. Let

$$\Delta_n = \left|\frac{1}{\pi}\int fg\,dx - (\frac{1}{2}a_0a_0' + \sum_1^n (a_ka_k' + b_kb_k'))\right| = \frac{1}{\pi}\left|\int (g - S_n(g))f\,dx\right| .$$

Here $S_n(g)$ denotes the n-th partial sum of the Fourier series of g. Then $g \in$ HBV implies that $S_n(g) \to g$ everywhere and $S_n(g)$ is uniformly bounded. Then the Lebesgue dominated convergence theorem implies $\Delta_n \to 0$.

On the other hand, if ΛBV is not contained in HBV, there is $\{a_n\} \searrow 0$ such that $\Sigma\, a_n/\lambda_n$ converges and $\Sigma\, a_n/n$ diverges. Let

$$g_n(x) = \begin{cases} a_i, & \frac{(2i-2)\pi}{n+1/2} < x < \frac{(2i-1)\pi}{n+1/2}, \quad i=1,\ldots,n+1 \\ 0, & \text{otherwise.} \end{cases}$$

ΛBV is a Banach space with norm $g \to |g(0)| + V_\Lambda(g)$. Hence

$$\|g_n\|_\Lambda = V_\Lambda(g_n) \le 2\sum_1^\infty a_i/\lambda_i = C < \infty$$

for every n. Then

$$\sup_x |S_n(g_n,x)| \geq |S_n(g_n,0)| > \frac{2}{\pi^2}\sum_1^{n+1} a_i/_i \neq O(1),$$

implying that for $\{P_n\}$, the sequence of continuous linear functionals on L defined by

$$P_n(f) = \int_0^{2\pi} fS_n(g_n)dx,$$

we have $\|P_n\| \neq O(1)$. Hence there is an $f_0 \in L$ such that $P_n(f_0) \neq O(1)$. Then

$$Q_n(g) = \int_0^{2\pi} f_0 S_n(g)dx$$

defines a sequence of continuous linear functionals on ΛBV and

$$\|Q_n\| \geq |Q_n(g_n)|/\|g_n\| \geq |P_n(f_0)|/C \neq O(1).$$

This implies that there is a $g_0 \in \Lambda BV$ such that

$$Q_n(g_0) = \int_0^{2\pi} f_0 S_n(g_0)dx \neq O(1) \quad \text{or} \quad \frac{1}{2}a_0a_0 + \sum_1^\infty (a_k a_k' + b_k b_k')$$

diverges.

We conclude with a few open problems.

1. Continuity in Λ-variation. Suppose $\lambda_n \nearrow \infty$ and $\Lambda^m = \{\lambda_{n+m}\}$, $n=1,2,\ldots$. Then $f \in \Lambda BV$ implies $f \in \Lambda^m BV$. If $V_{\Lambda^m}(f) \to 0$ as $m \to \infty$, f is said to be <u>continuous in Λ-variation</u>. This notion was of use in our study of (C,β)-summability of Fourier series, $-1 \leq \beta \leq 0$. Our question is: does $f \in \Lambda BV$ imply that f is continuous in Λ-variation? We conjecture that the answer is negative.

2. Positive and negative Λ-variations. We have defined the positive and negative Λ-variations of $f \in \Lambda BV$ and investigated their continuity properties. The positive Λ-variation is defined for $x \in [a,b]$ by

$$V_\Lambda^+(f,x) = \sup\{\Sigma(f(b_n)-f(a_n))/\lambda_n\},$$

the supremum being extended over collections of non-overlapping intervals (a_n,b_n) for which $f(b_n)-f(a_n) > 0$. The negative variation is analogously defined. Our question is: to what extent do V^+ and V^- determine f? A more explicit question, which would lead to others if the answer were negative, is: can two different functions have the same V^+ and V^-?

3. <u>Garsia-Sawyer class</u>. Suppose f has only simple discontinuities and complete its graph by adjoining a vertical line segment between the upper and lower limits at each point of discontinuity. Then we can extend the definition of the Banach indicatrix of f by setting $n(y) = n(f,y)$ equal to the cardinality of the set of x for which (x,y) is in the completed graph of f is this set is finite and equal to ∞ elsewhere. Let $A = \inf f$, $B = \sup f$. If $L(x)$ is an increasing positive function with $L(n) \sim \sum_1^n 1/\lambda_k$, then

$$\int_A^B L(n(y))dy < \infty$$

implies that $f \in \Lambda BV$. The Garsia-Sawyer class consists of those functions for which $\int \log^{+}(n(y))dy < \infty$, and this class is then contained in HBV. However, the Garsia-Sawyer class is not closed under addition, so it is a proper subclass of HBV. Let $\overline{GS}$ be the closed linear span of the Garsia-Sawyer class in HBV. Our question is: is $\overline{GS} = HBV$?

4. <u>Revision</u>. Let us call f a <u>revision</u> of g if $n(f,y) \equiv n(g,y)$. Clearly this is an equivalence relation. These are obvious operations on a function which result in revisions, e.g., composition with a homeomorphism. We ask: can we describe the operations by which all the revisions of a function may be obtained? (It may be reasonable to assume the functions to be continuous.)

5. <u>Multipliers</u>. Let $\hat{f}(k)$ denote the k-th Fourier coefficient of f. If F and G are two classes of functions, a complex valued function φ on Z is a <u>multiplier</u> of <u>type</u> $(\underline{F},\underline{G})$ if $f \in F$ implies that there is an $h \in G$ such that $\hat{h}(k) = \varphi(k)\hat{f}(k)$ for every $k \in Z$. Many obvious (and difficult) questions can be posed concerning multipliers and ΛBV and HBV. Are there interesting function classes F and G such that $\hat{f}$, for each $f \in \Lambda BV$, is a multiplier of type (F,G)? Can we characterize the multipliers of types $(\Lambda BV, G)$ and $(F, \Lambda BV)$ for any significant choices of F, G, and Λ?

6. <u>Ordered ΛBV</u>. A function f on I is of <u>ordered Λ-bounded variation</u> ($O\Lambda BV$) if

$$\sup\{\Sigma |f(a_n)-f(b_n)|/\lambda_n\} < \infty,$$

where the supremum is extended over all collections of intervals (a_n, b_n) for which either $b_n \le a_{n+1}$ for each n or $b_{n+1} \le a_n$ for each n. Clearly $O\Lambda BV \supseteq \Lambda BV$. Is this inclusion proper? If so, what properties of ΛBV carry over to $O\Lambda BV$. What can be said of the Fourier series of functions of $O\Lambda BV$ and, in particular, of $OHBV$?

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